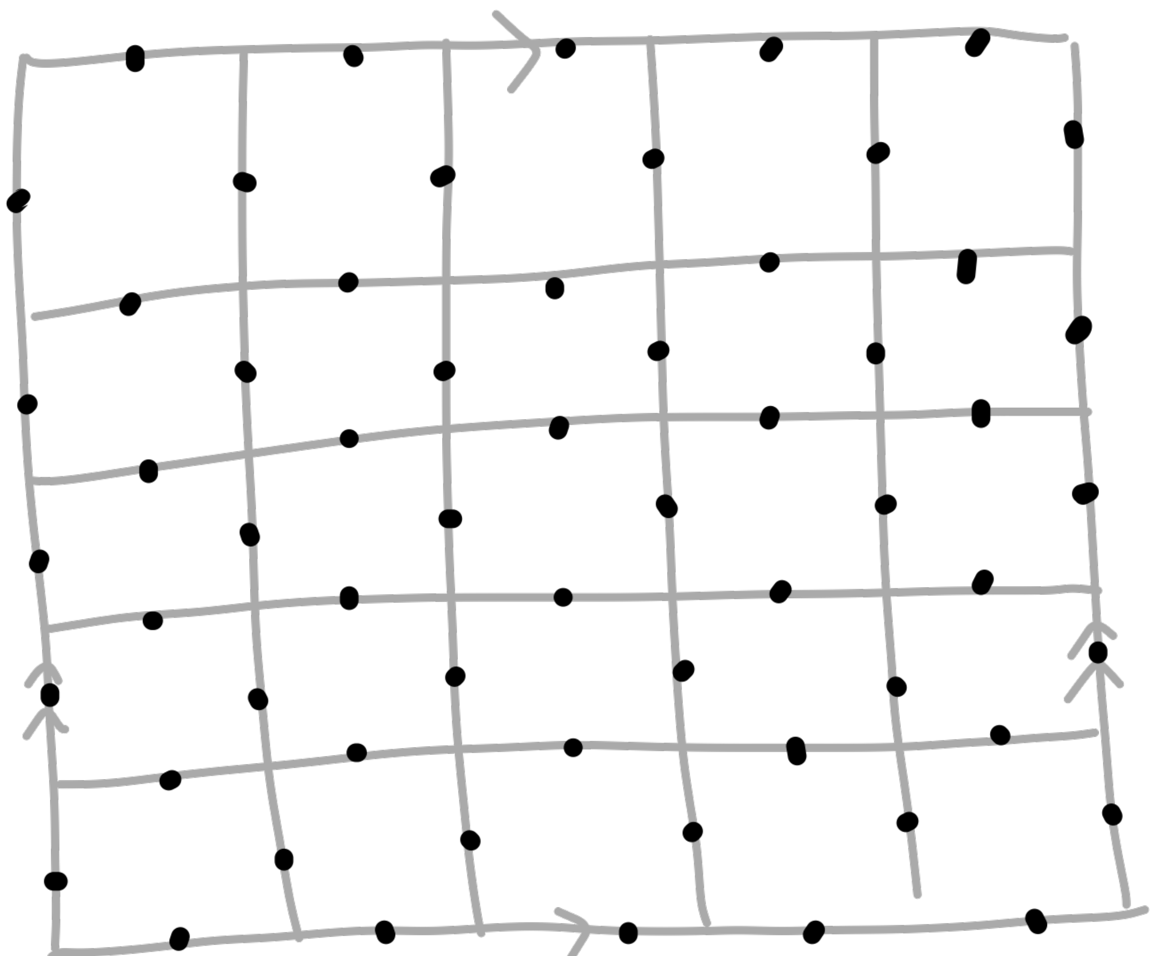
Enriched topological codes

101 applications of the tube algebra

Daniel Barter, Jacob Bridgeman, Corey Jones

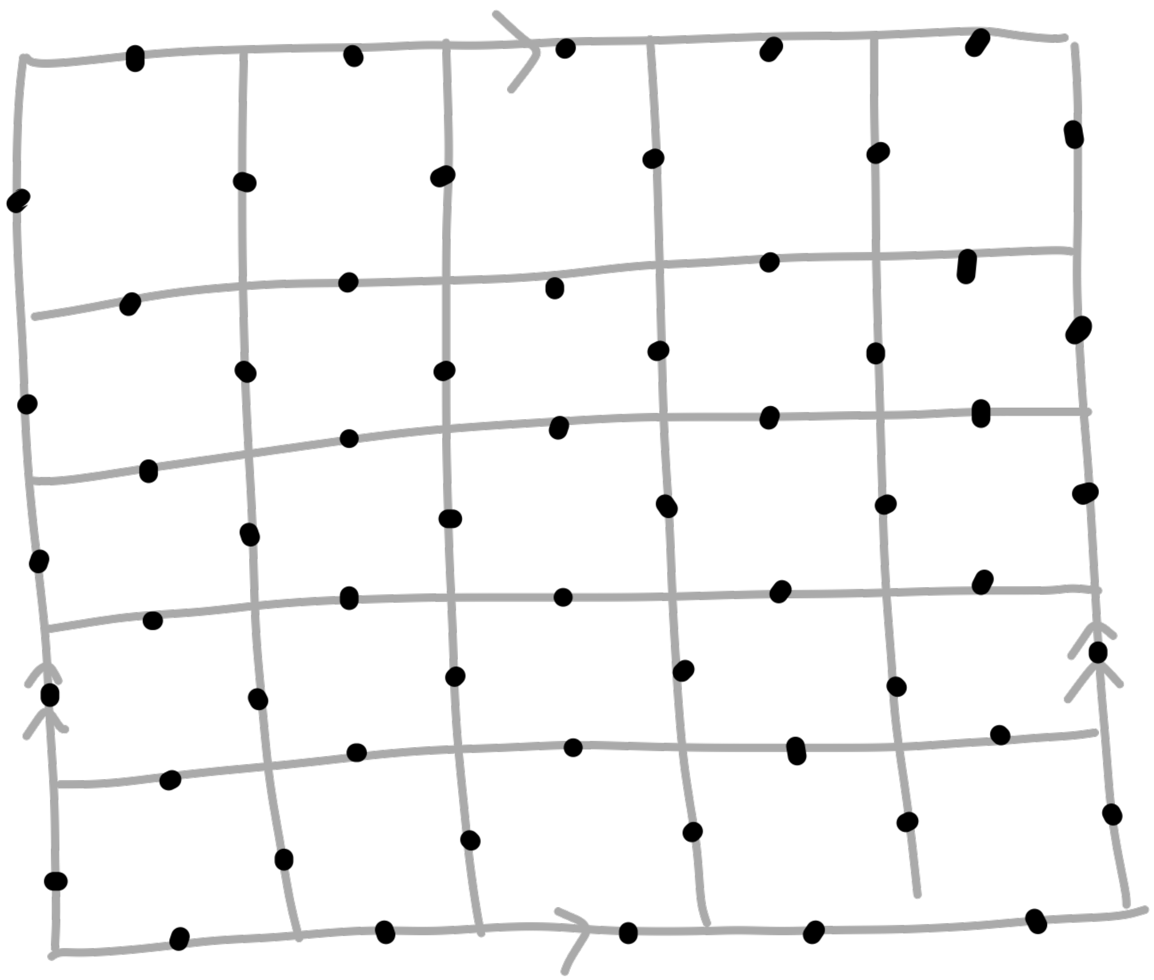


$\bullet = \mathbb{C}^2 = \mathrm{span}\{|0\rangle, |1\rangle\}$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$XZ = -ZX$

$H = -\sum_{f} X\,X\,X\,X \;-\; \sum_{v} Z\,Z\,Z\,Z$



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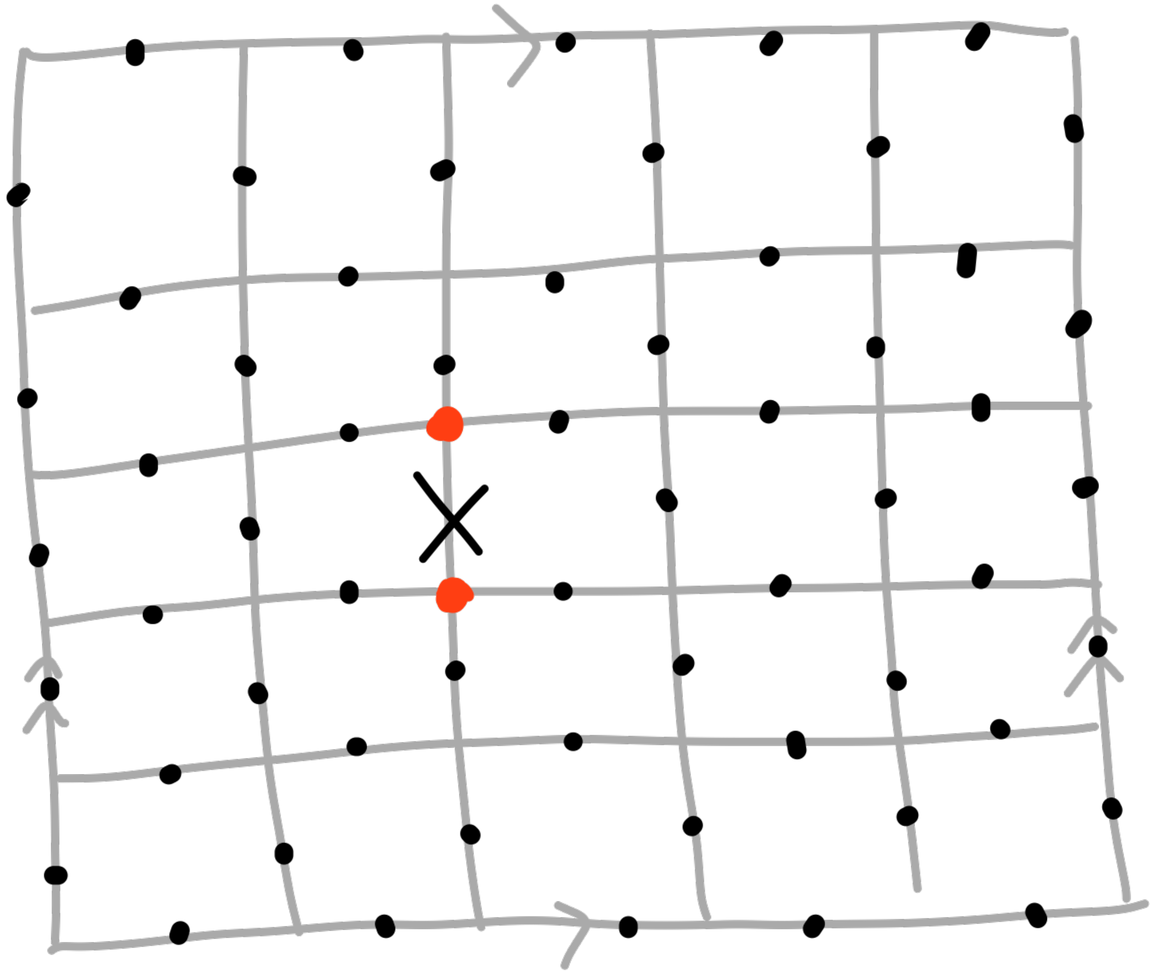

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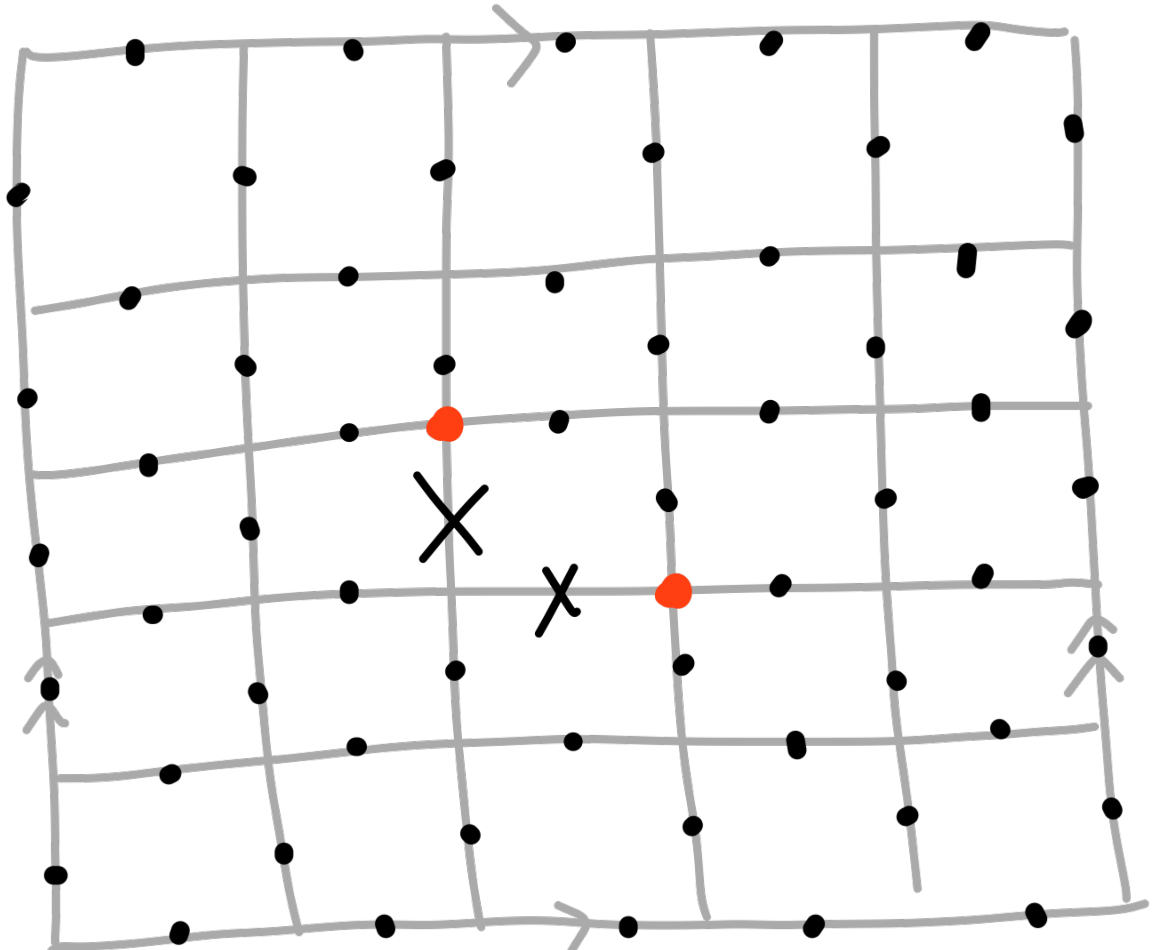

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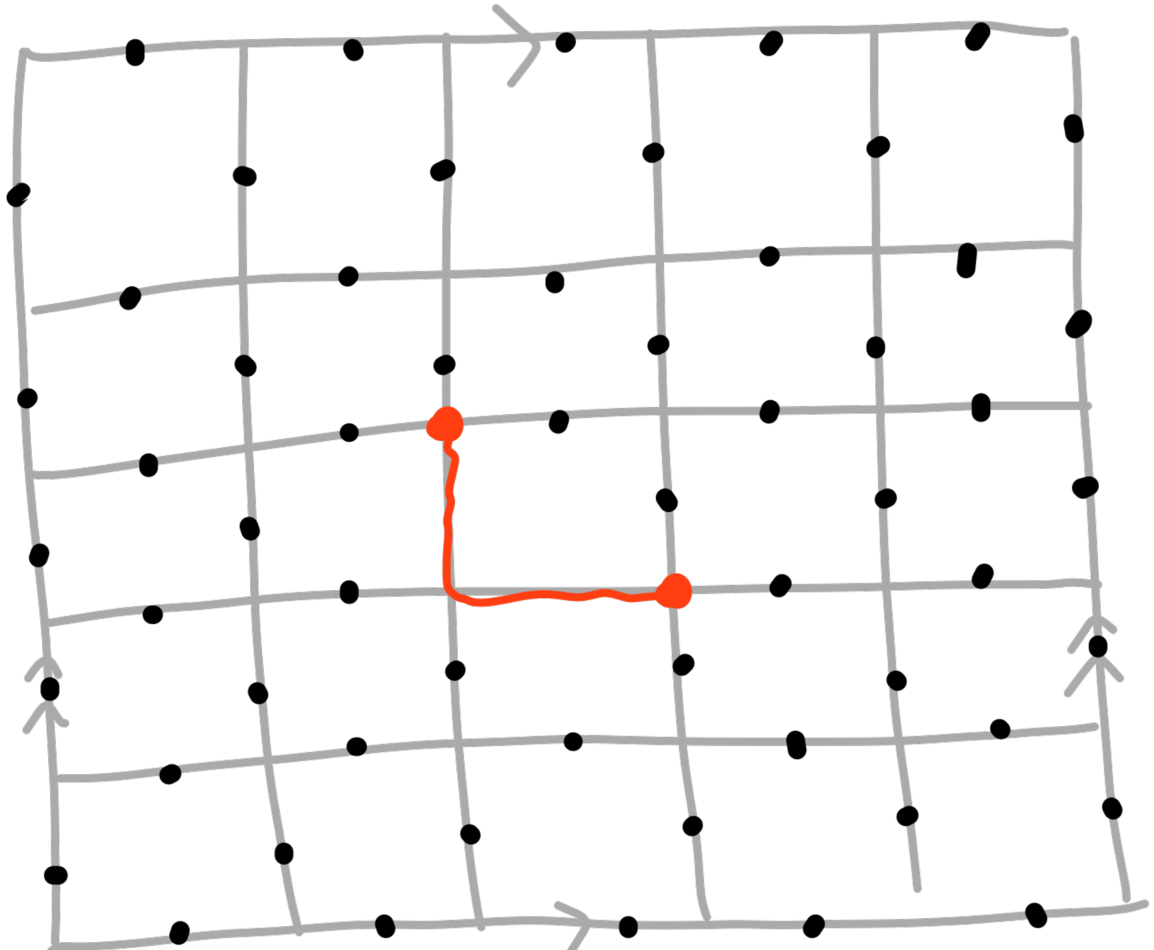

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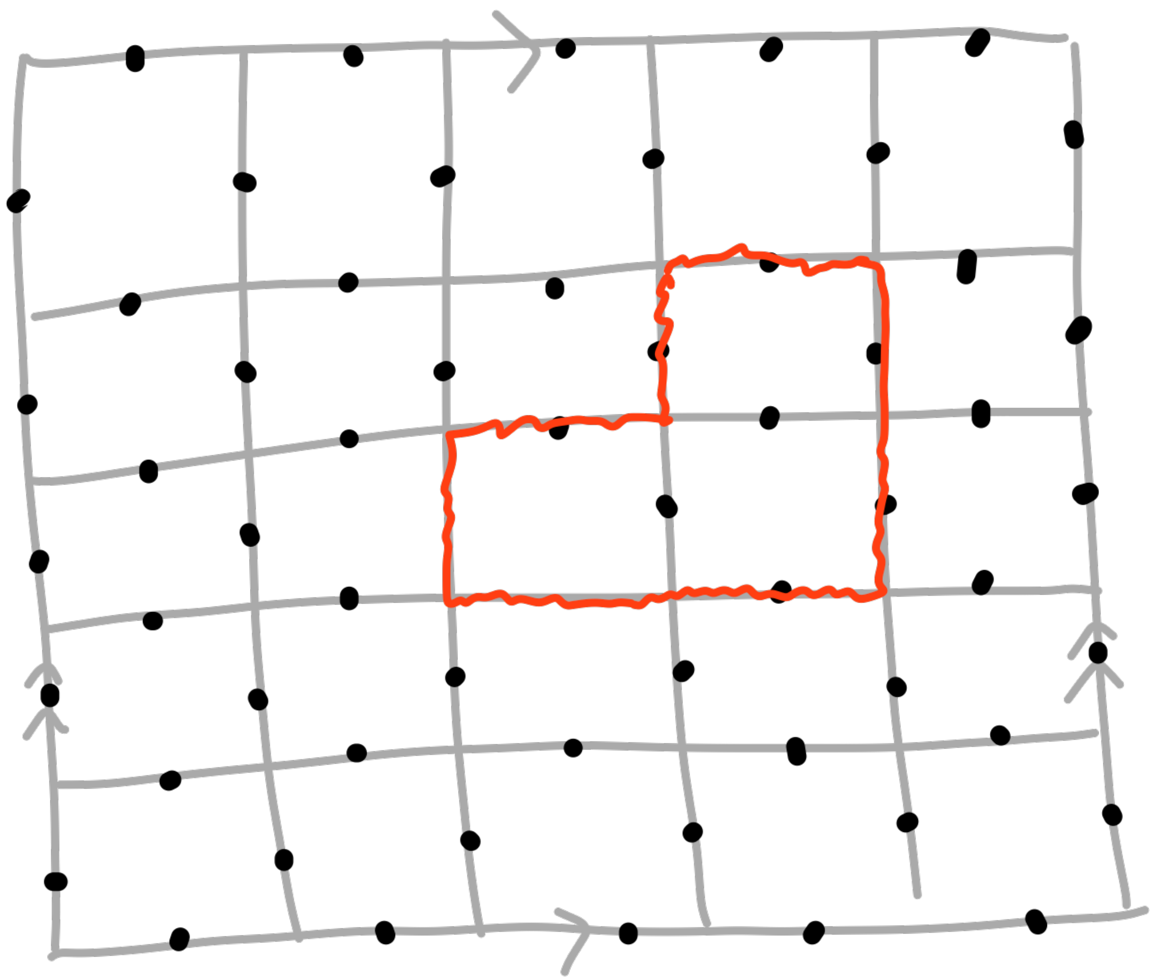

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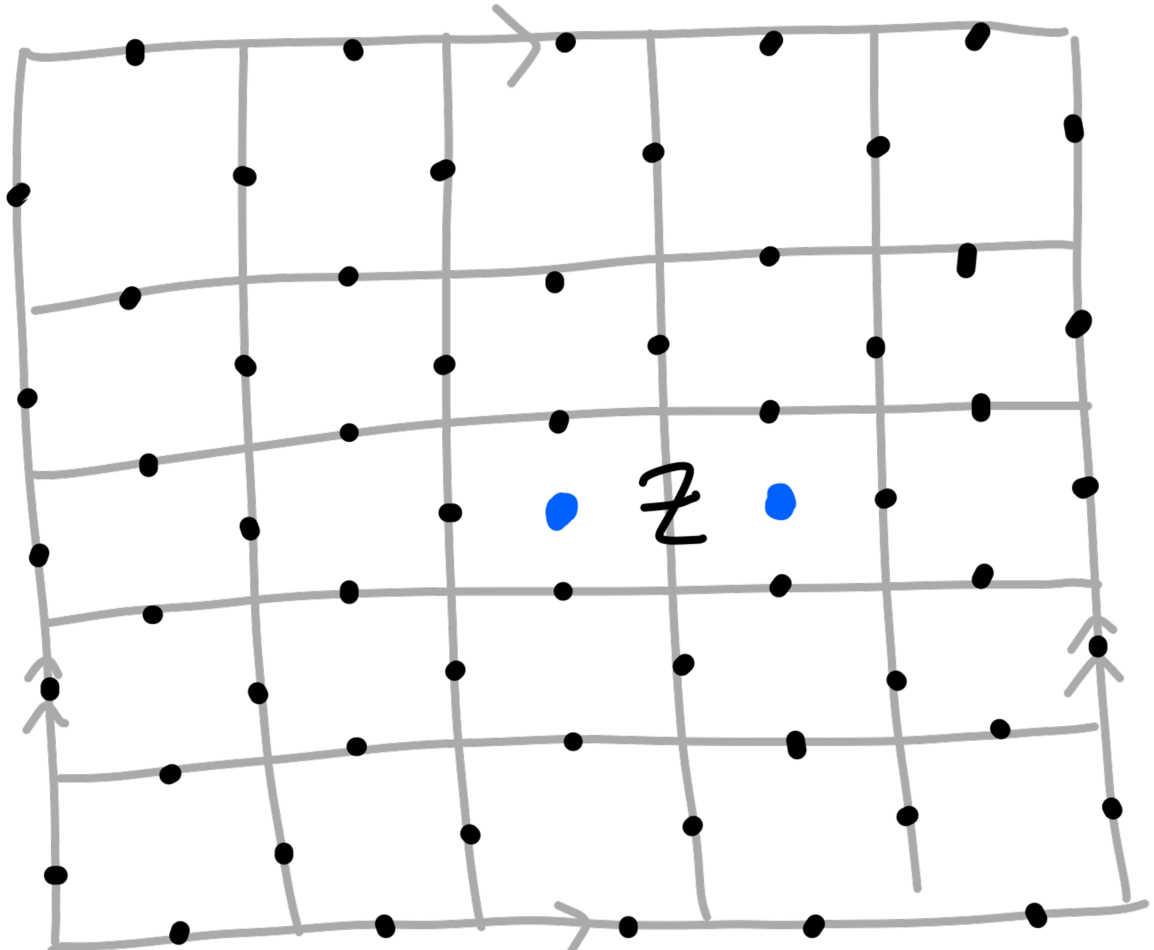
$XZ = -ZX$

$H = -\sum_{f} X X X X - \sum_{v} Z Z Z Z$

$X^2 = 1$



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$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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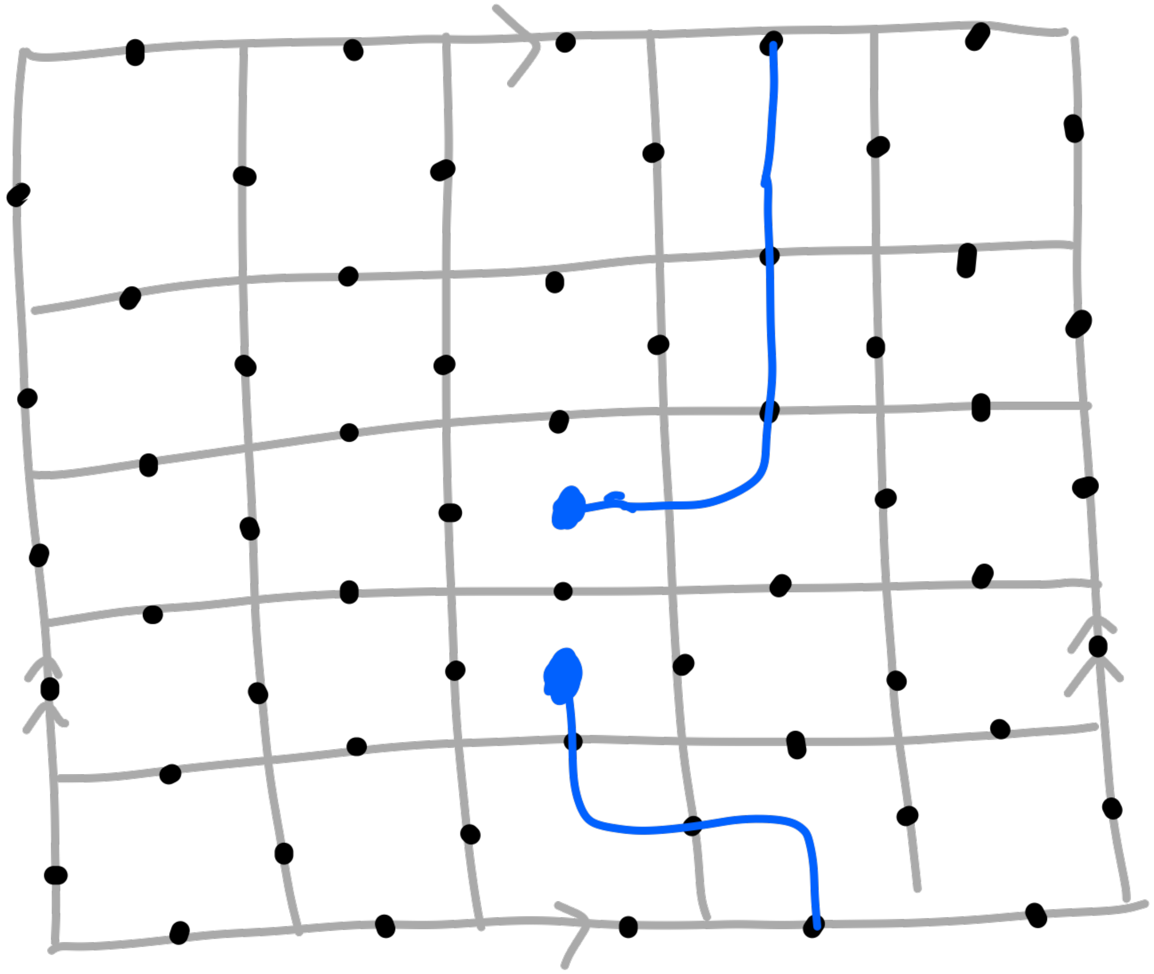
$$H = -\sum_f \begin{matrix} & X & \\ X & f & X \\ & X & \end{matrix} \;-\; \sum_v \begin{matrix} & Z & \\ Z & v & Z \\ & Z & \end{matrix}$$



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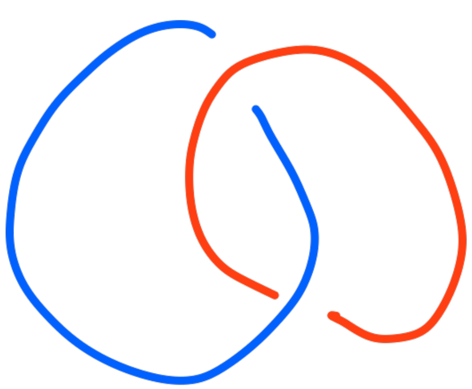
$$H = -\sum_f \prod_{e \in f} X \; - \sum_v \prod_{e \ni v} Z$$

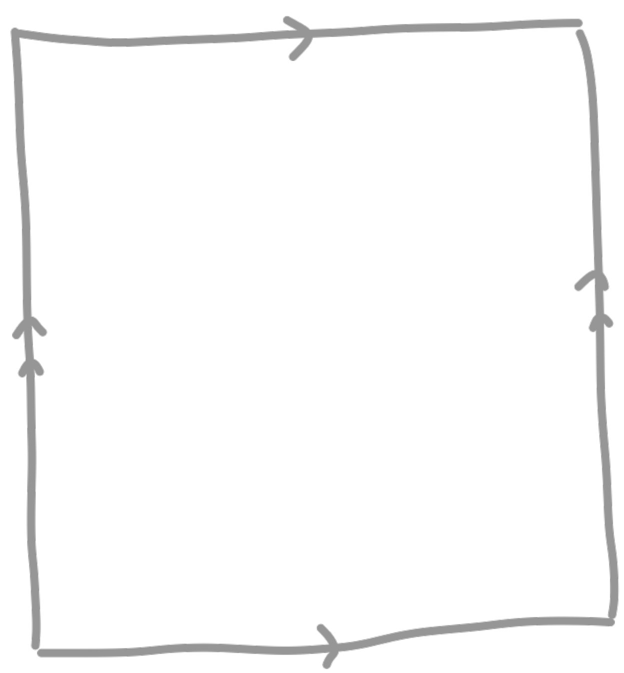


$= -1$

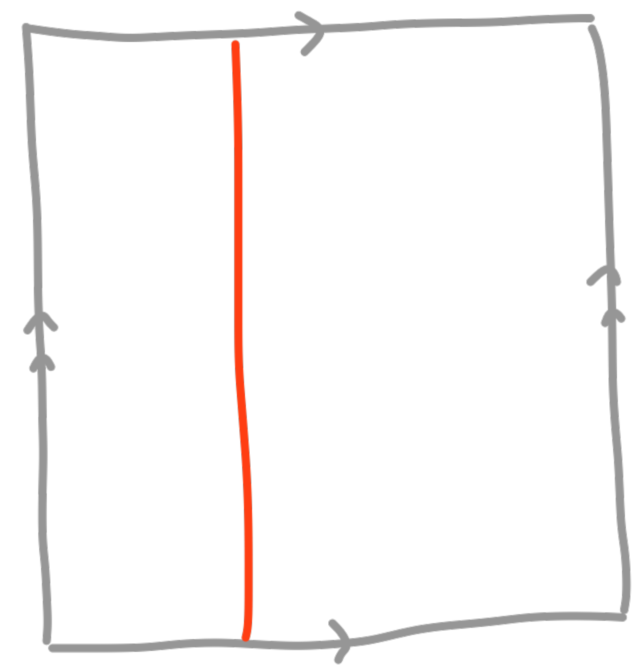
$\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}})$

4 ground states :

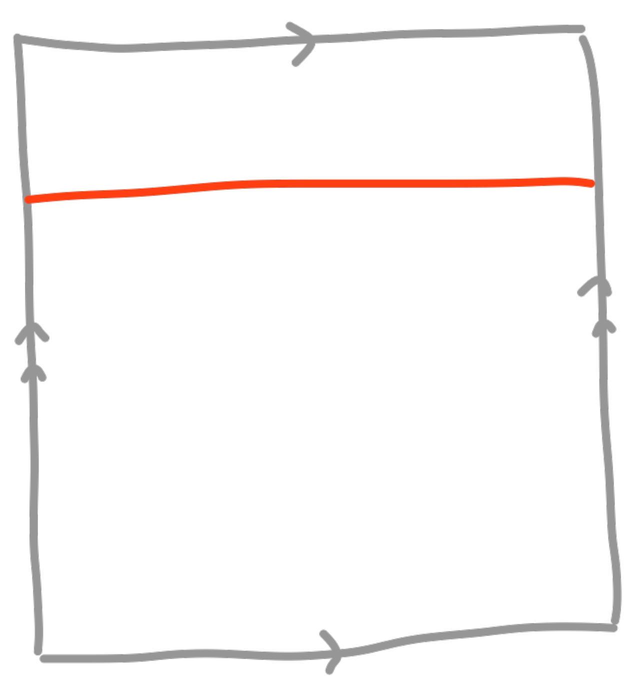
$|00\rangle \rightarrow$



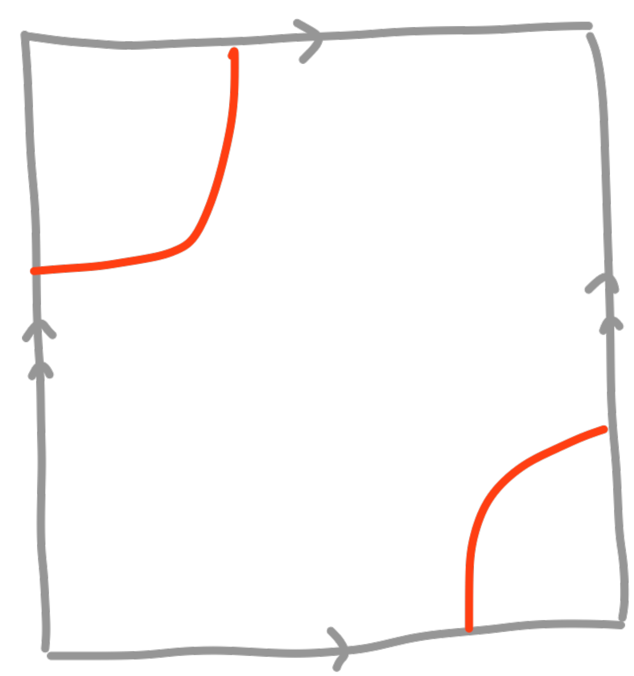
$|01\rangle \rightarrow$

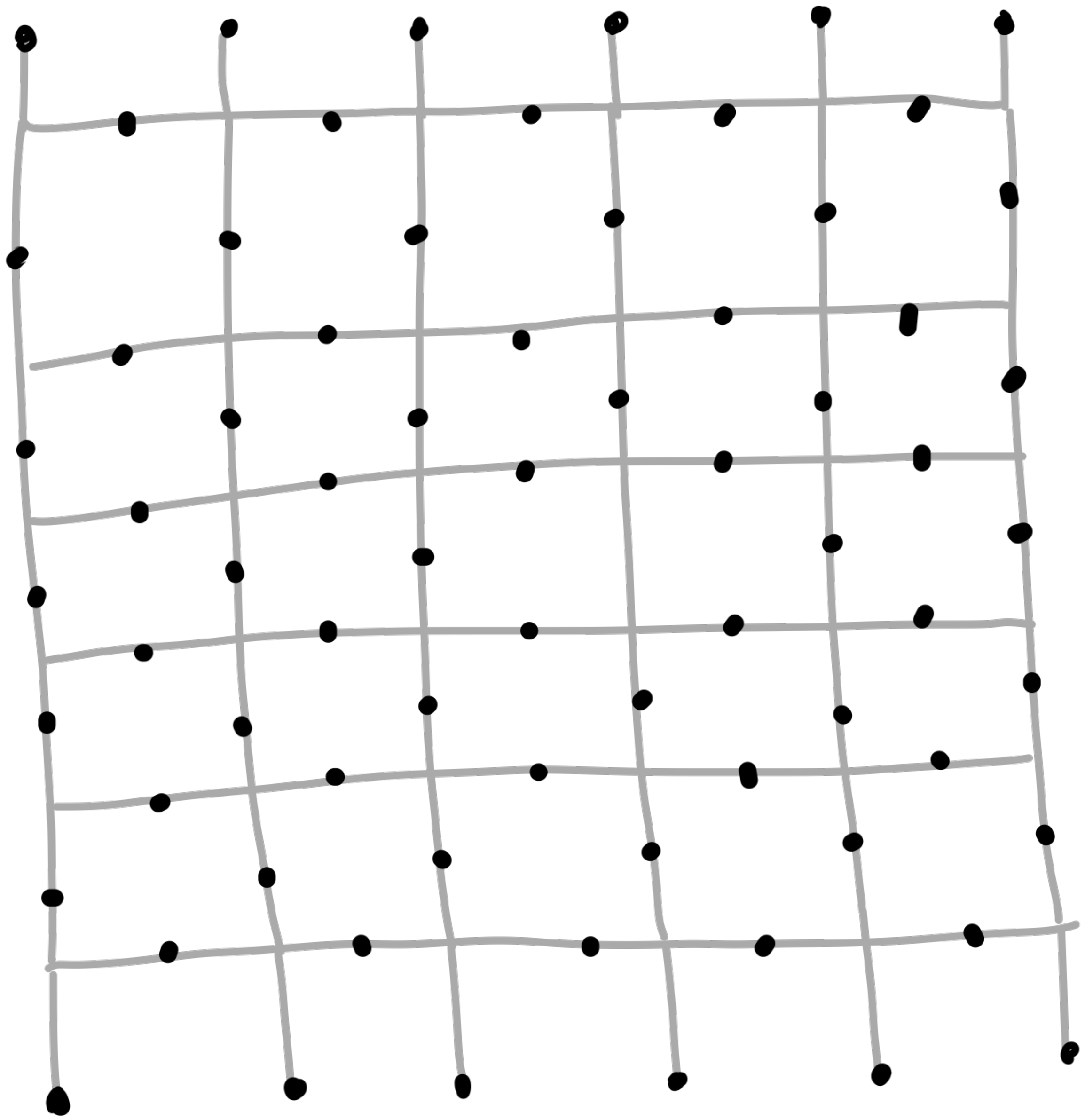


$|10\rangle \rightarrow$



$|11\rangle \rightarrow$





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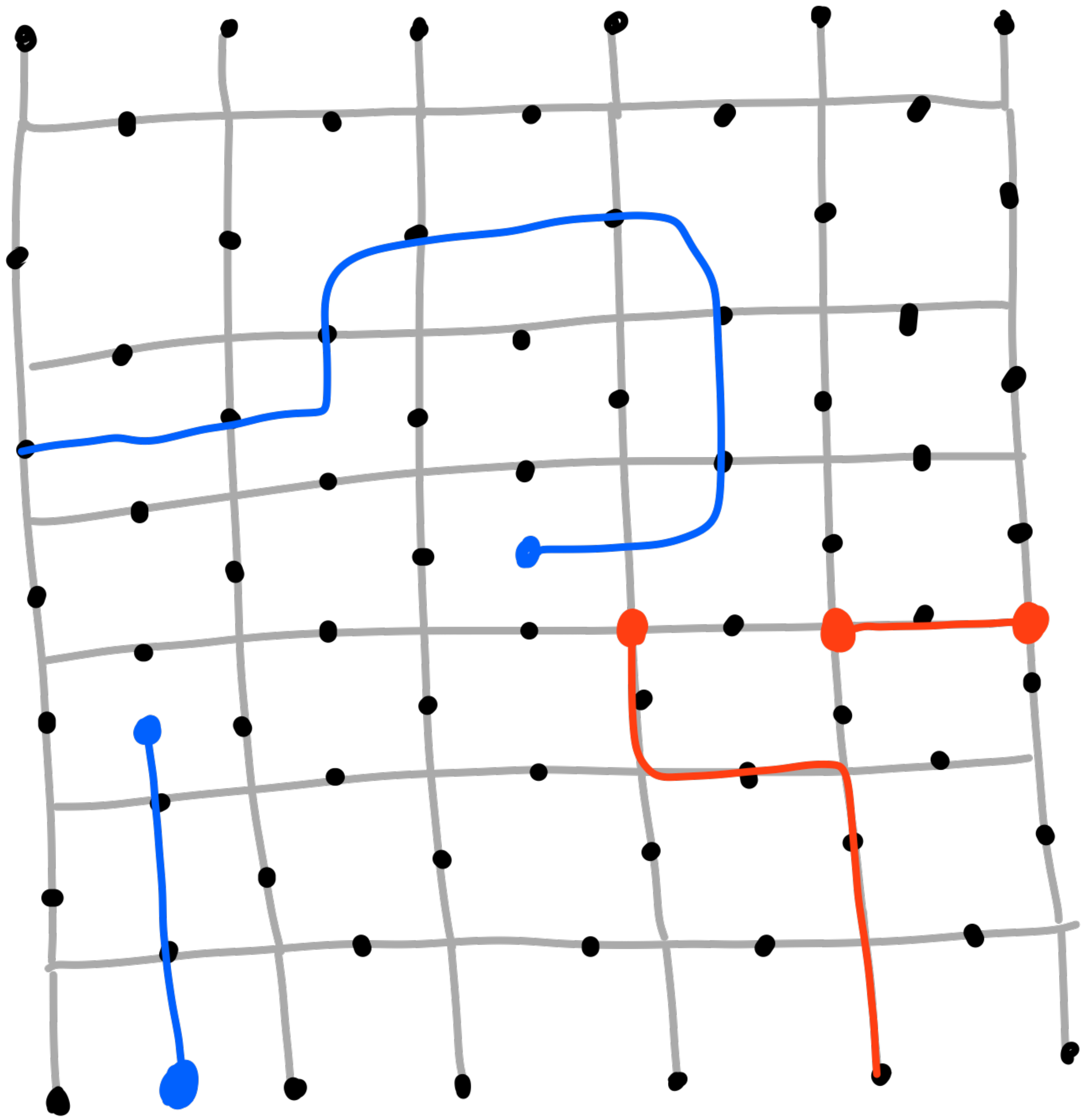
$$H = -\sum_f \prod_{\partial f} X \; - \sum_v \prod_{v} Z \; - \sum X X X \; - \sum Z Z Z$$



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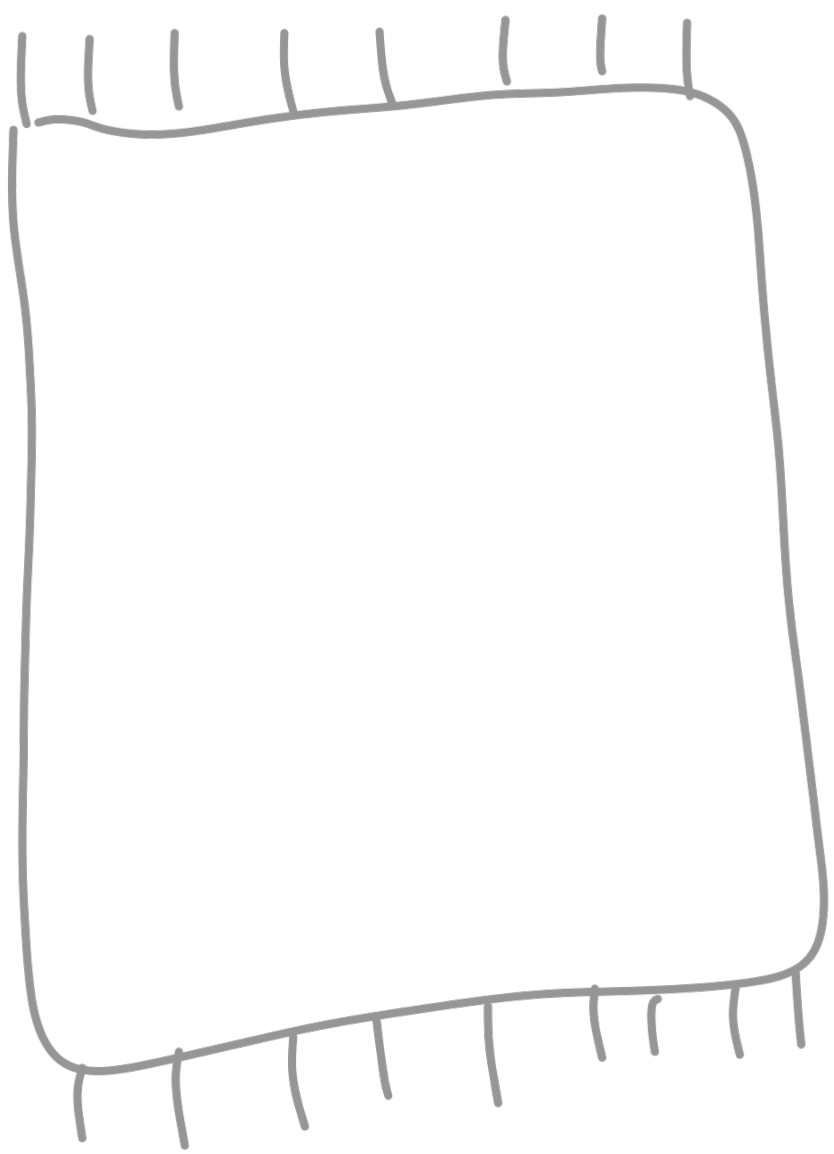
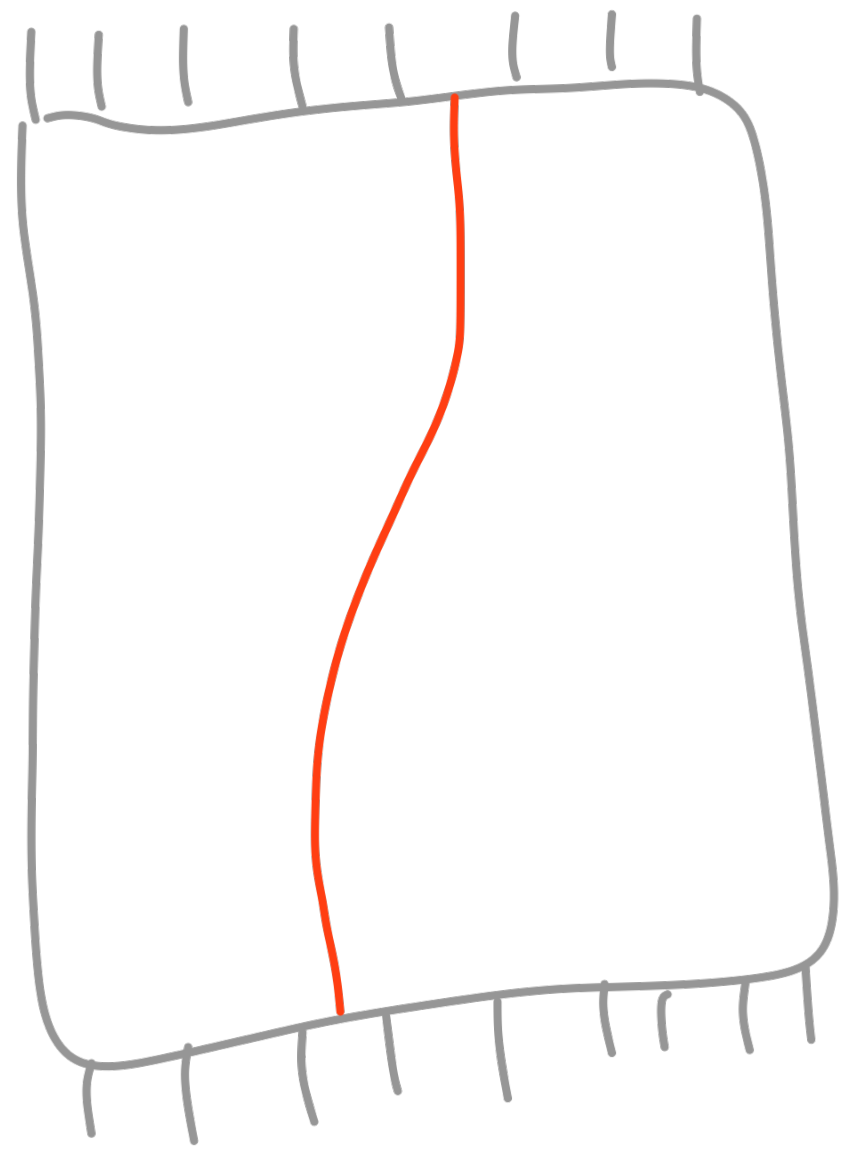
$XZ = -ZX$

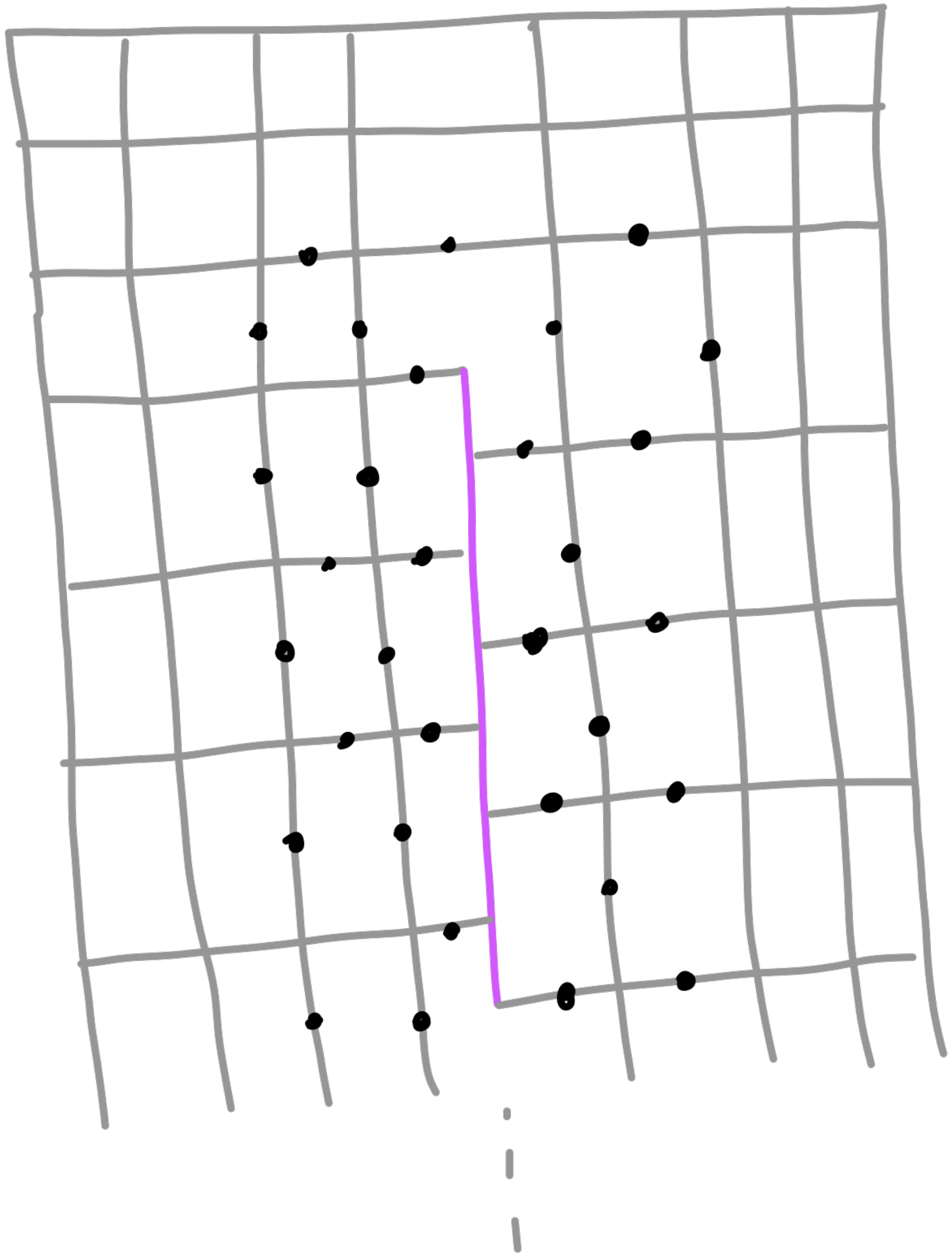
$H = -\sum_f \; X \, X \, X \, X \; - \sum_v \; Z \, Z \, Z \, Z$

$- \sum \; X \, X \, X \; - \sum \; Z \, Z \, Z$

Only 2 ground states:







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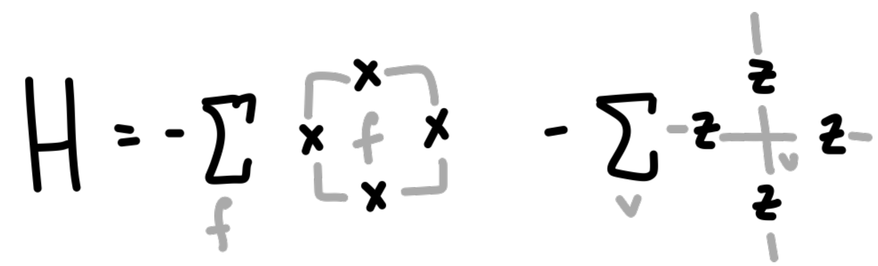

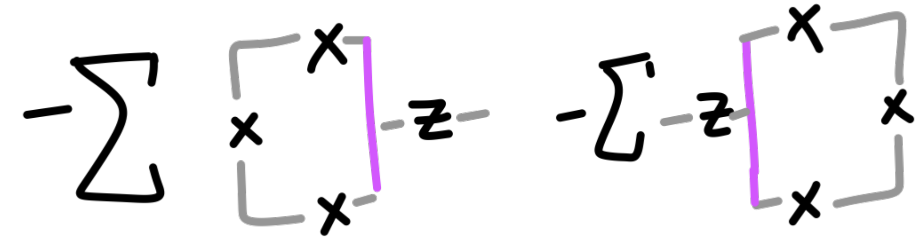

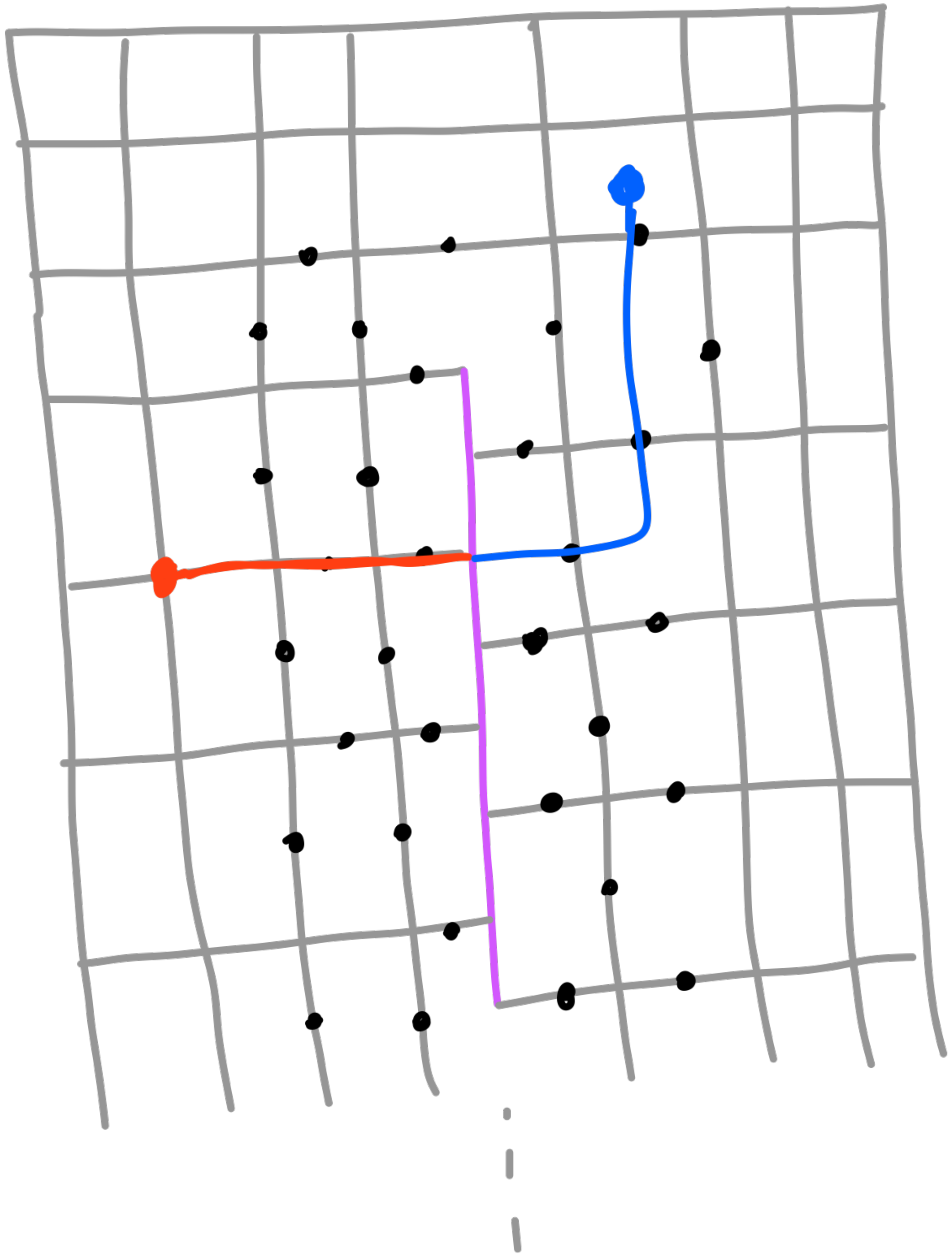

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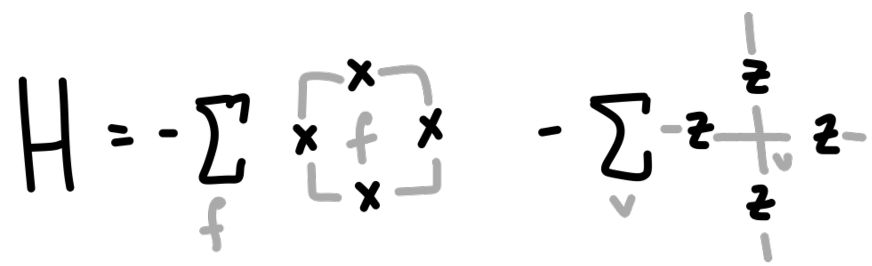
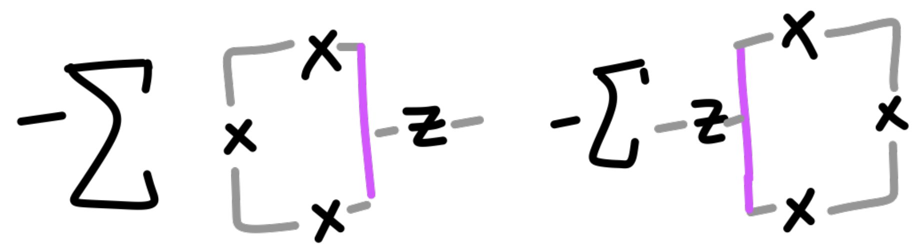
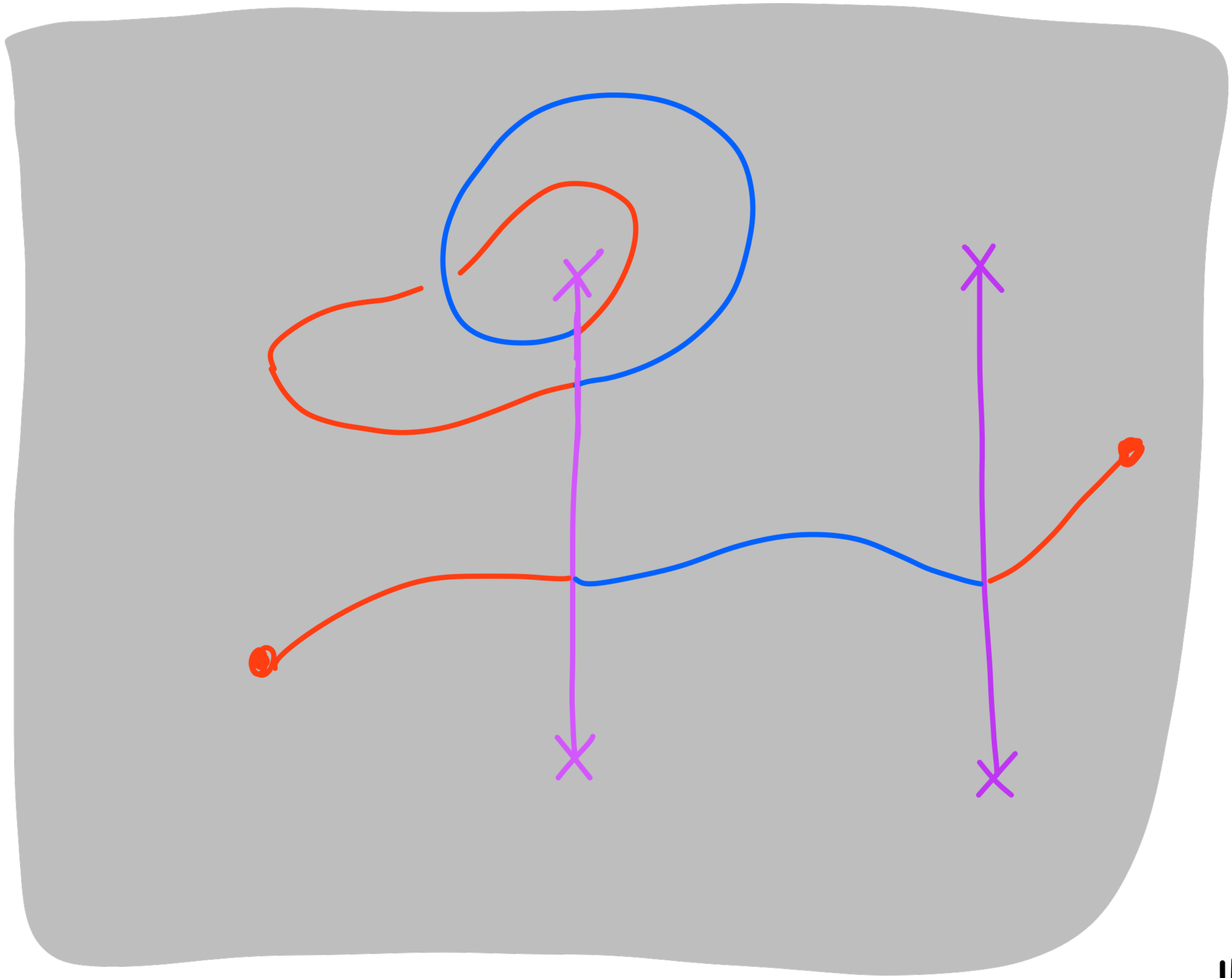
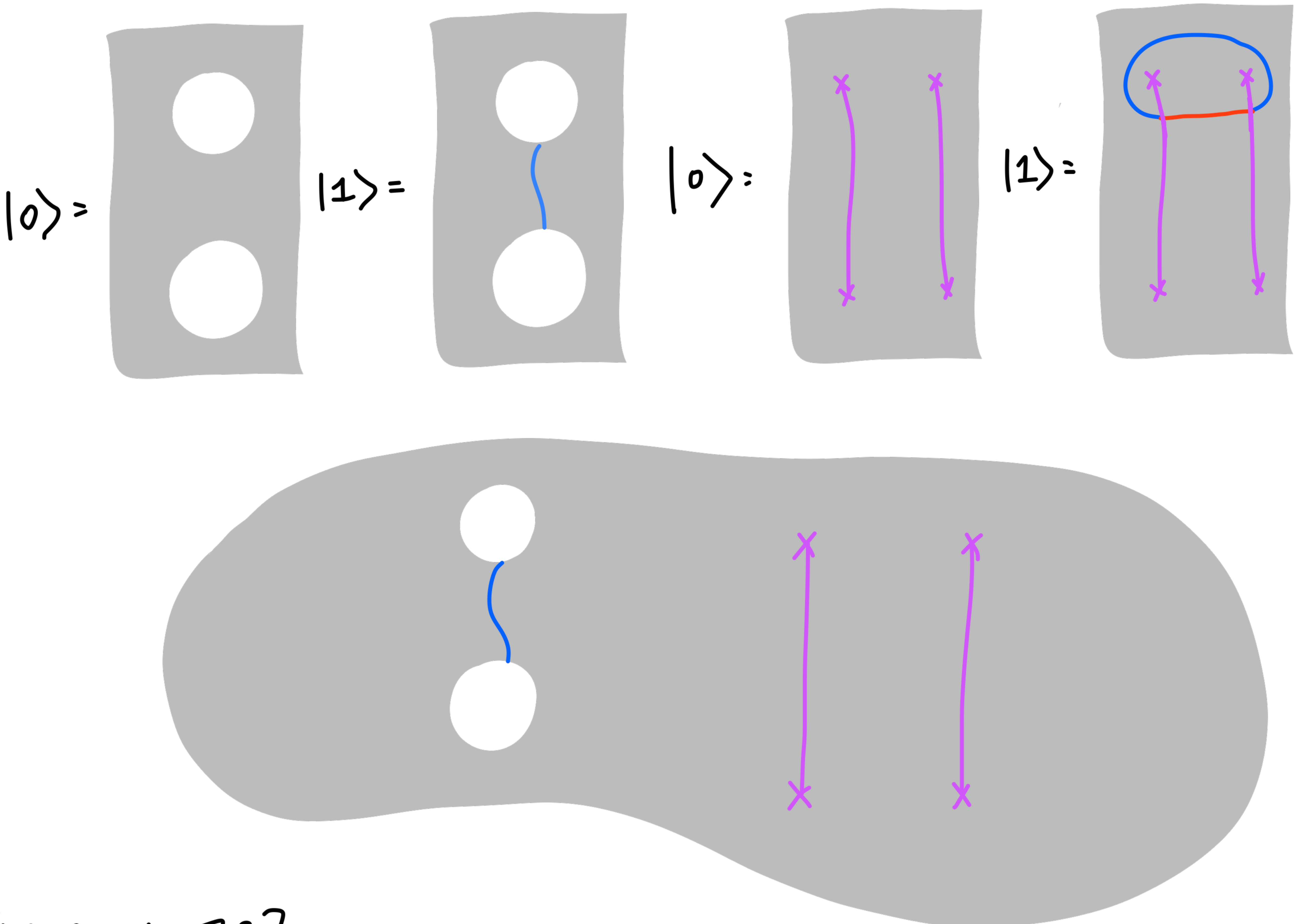






= Invertible domain wall



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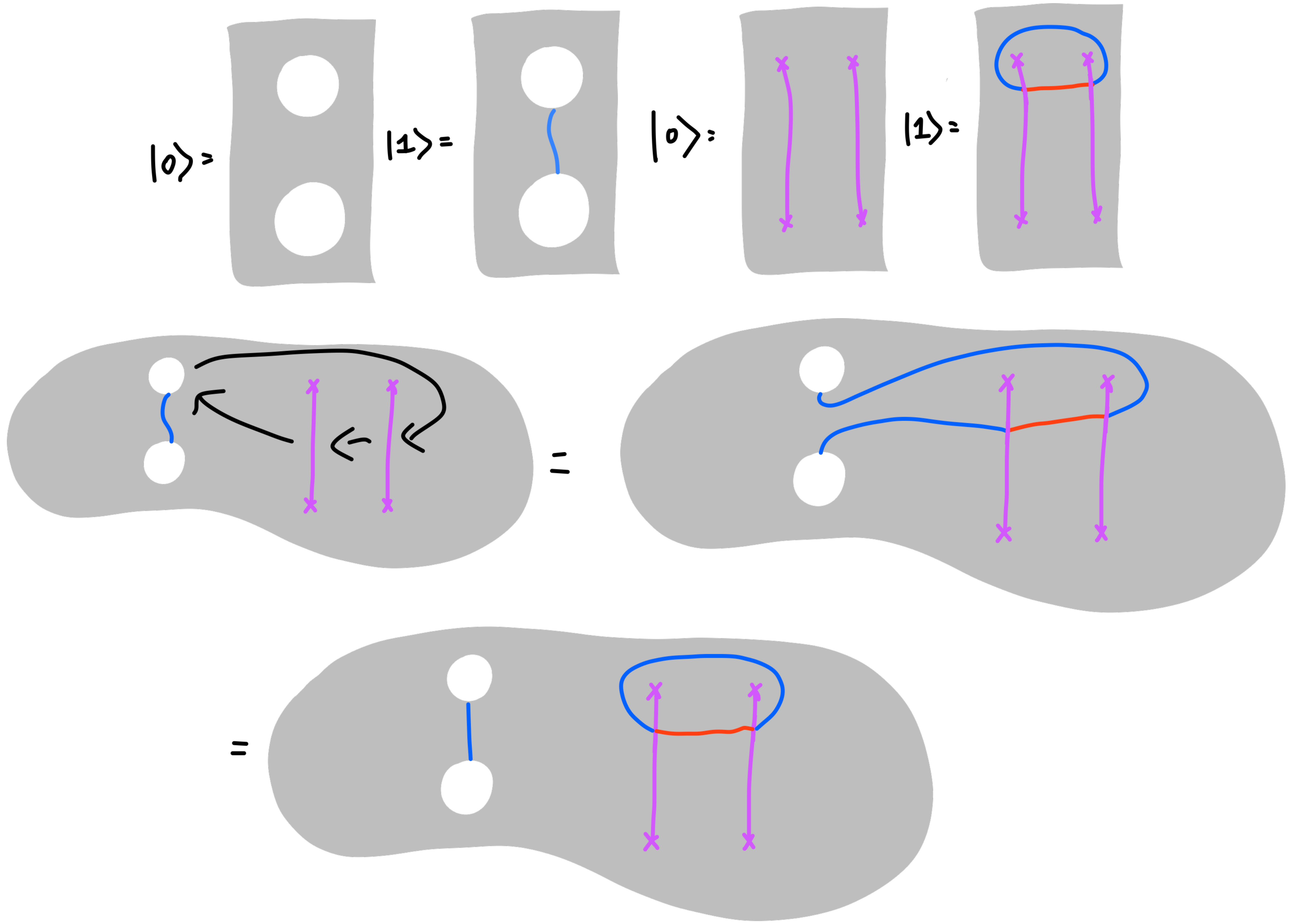


|0⟩ =
|1⟩ =
|0⟩ =
|1⟩ =
=
=

"Enriching" topological code can be useful for quantum computing

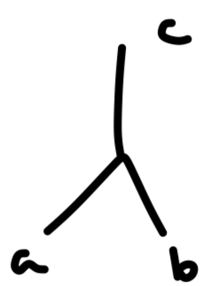
Although abstract classifications exist, need data to design QC schemes

Fusion cat $\mathcal{C}$

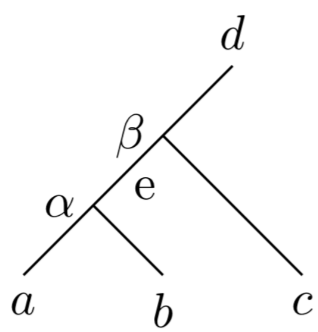
$$\Big\downarrow \text{(Levin Wen construction)}$$

Lattice model with excitations described by $Z(\mathcal{C})$

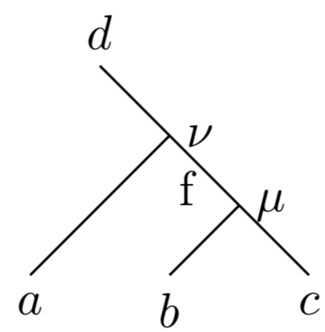
Skeletal fusion category



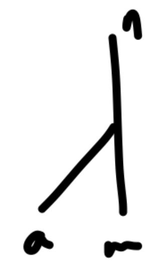
$= \mathrm{Hom}(a \otimes b, c)$



$= \sum_{\mu\nu} F^{\mu\nu}_{\alpha\beta}$

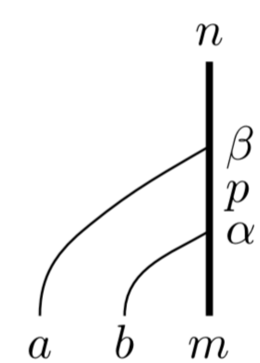


Domain wall = Bimodule category

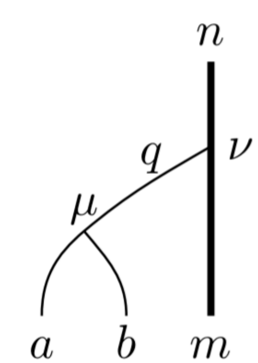


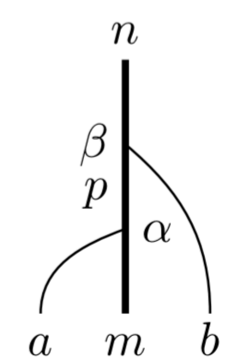
$= \mathrm{Hom}(a \triangleright m, n)$

$= \mathrm{Hom}(m \triangleleft b, n)$

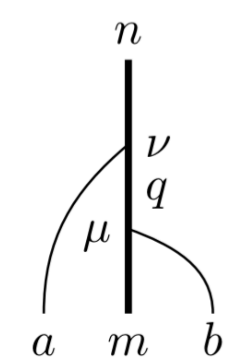


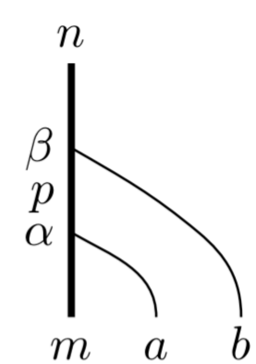
$= \sum_{\mu\nu} L^{\mu\nu}_{\alpha\beta}$



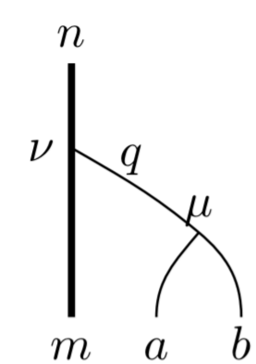


$= \sum_{\mu\nu} C^{\mu\nu}_{\alpha\beta}$





$= \sum_{\mu\nu} R^{\mu\nu}_{\alpha\beta}$



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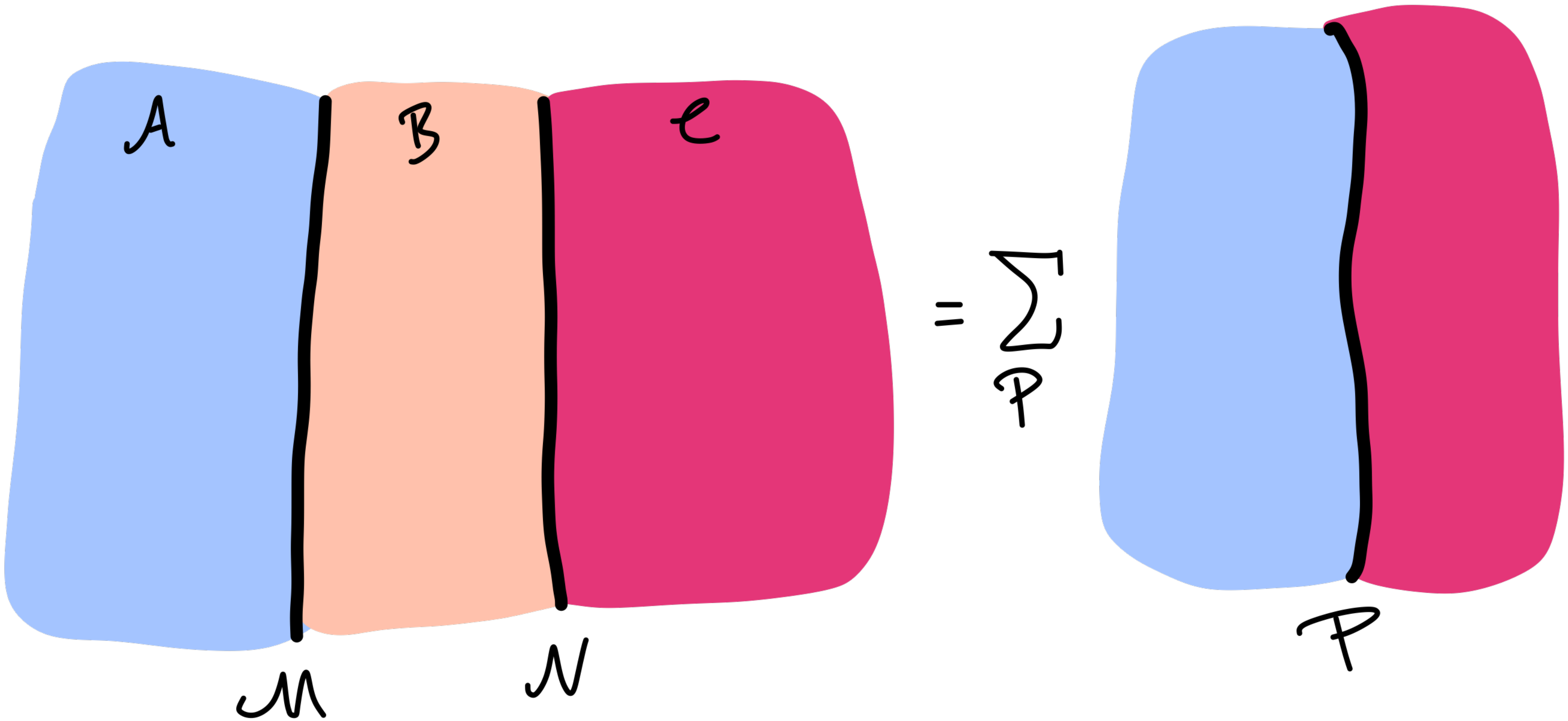
Fusing domain walls

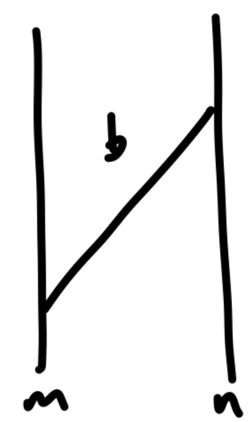


$$\mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \cong \sum_{\mathcal{P}} \mathcal{P}$$

$$\mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \cong \mathrm{Kar}\left(\mathcal{L}ad_{\mathcal{B}}(\mathcal{M},\mathcal{N})\right)$$

$$\mathrm{obj}\left(\mathcal{L}ad_{\mathcal{B}}(\mathcal{M},\mathcal{N})\right) = (m, n)$$

morphisms



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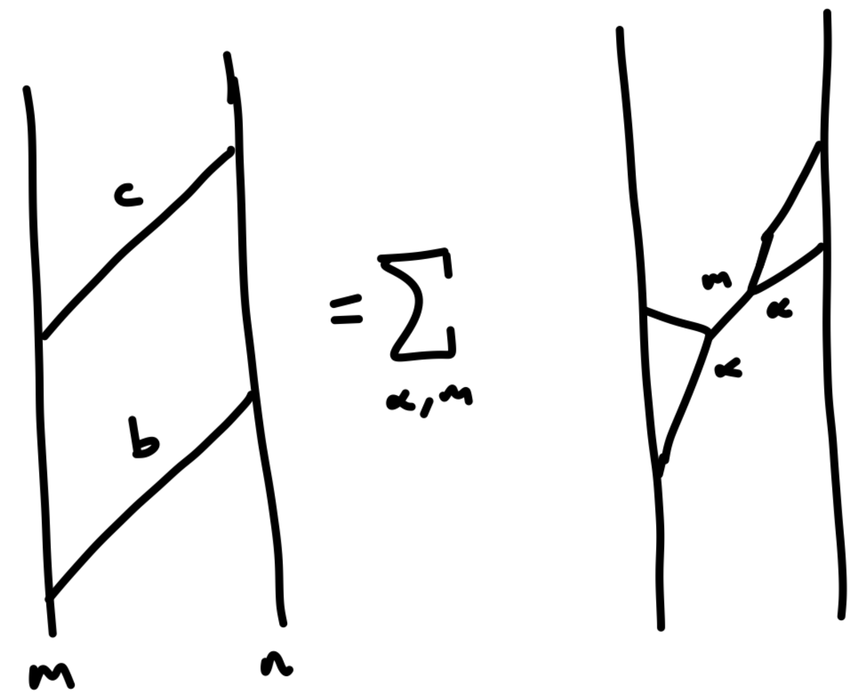
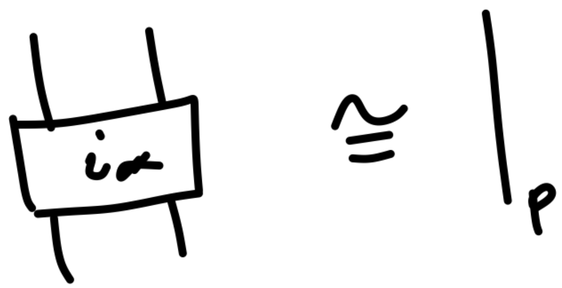


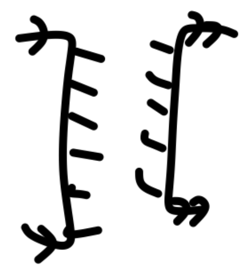
(Secretly a tube category)

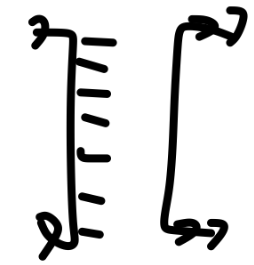
→ Iso classes of simples represented by idempotents

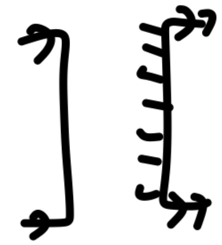


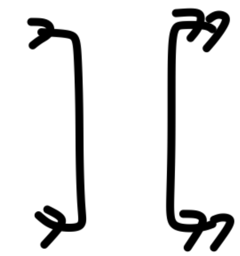
Identify left & right $\mathcal{C}$ action

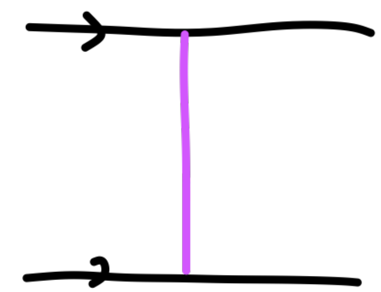
$\otimes_{\mathbf{Vec}(\mathbb{Z}/p\mathbb{Z})}$	T	L	R	F_0	X_l	F_r
T	$p\cdot T$	T	$p\cdot R$	R	T	R
L	$p\cdot L$	L	$p\cdot F_0$	F_0	L	F_0
R	T	$p\cdot T$	R	$p\cdot R$	R	T
F_0	L	$p\cdot L$	F_0	$p\cdot F_0$	F_0	L
X_k	T	L	R	F_0	X_{kl}	F_{k-1_r}
F_q	L	T	F_0	R	F_{ql}	X_{q-1_r}

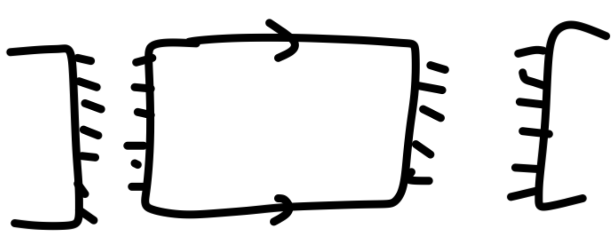
$T =$ 

$L =$ 

$R =$ 

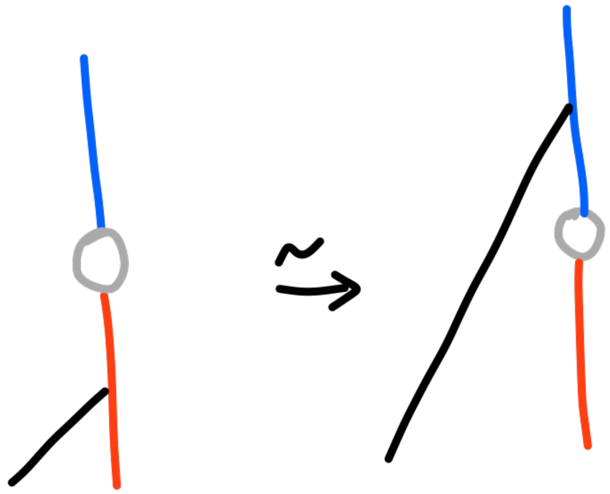
$F_0 =$ 

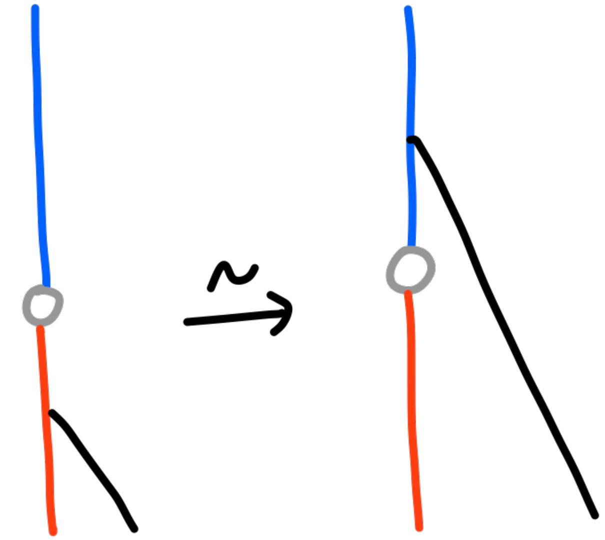
$F_1 =$ 

$T \otimes_{\mathbb{Z}/2} T$: 

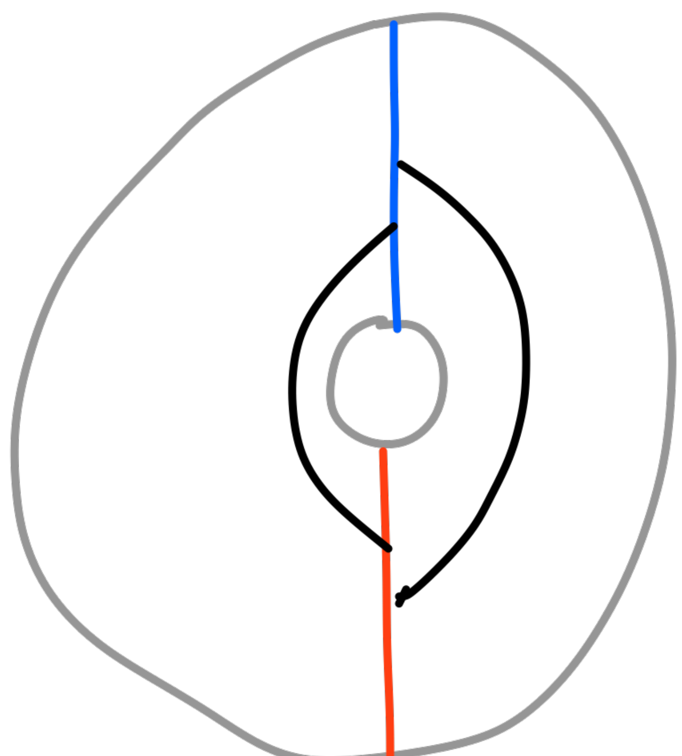
↑ 2 ground states

Domain wall excitations = Bimodule functors

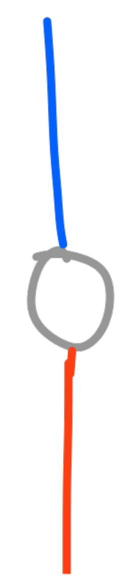




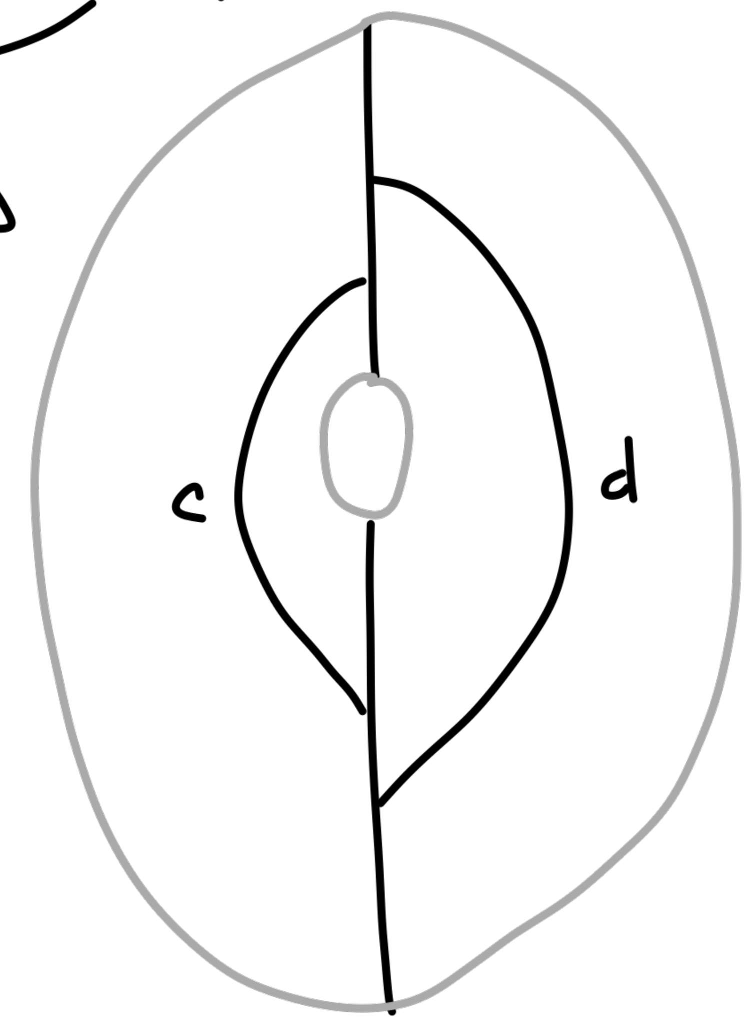
Generalised tube algebra



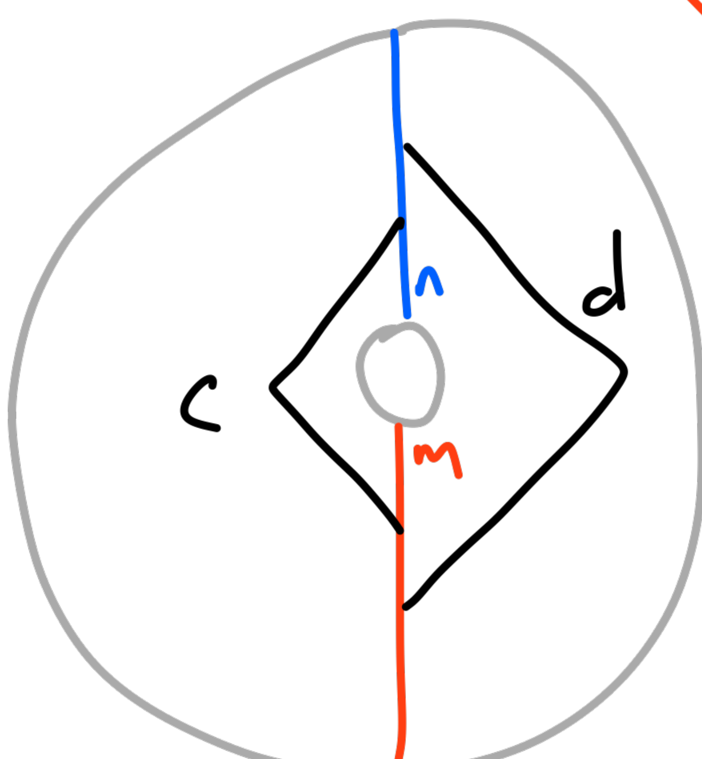
acts on



$$Z(\mathcal{C}) \cong \mathrm{Kar}(\underbrace{\mathrm{Tub}_{\mathcal{C},\mathcal{C}}(\mathcal{C},\mathcal{C})}))$$



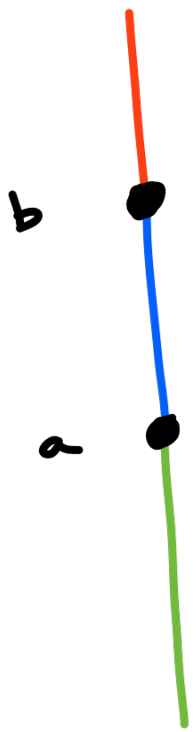
$$Z_{\mathcal{C},\mathcal{D}}(\mathcal{M},\mathcal{N}) \cong \mathrm{Kar}(\underbrace{\mathrm{Tub}_{\mathcal{C},\mathcal{D}}(\mathcal{M},\mathcal{N})}))$$



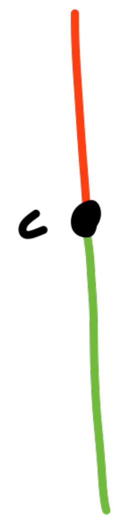
find idempotents

Vertical fusion = functor composition

$\mathrm{Vec}(\mathbb{Z}/3) \curvearrowright \mathcal{M} \curvearrowleft \mathrm{Vec}(S_3)$



$\simeq \sum_c N_{ab}^c$



End$(\mathcal{M})$:

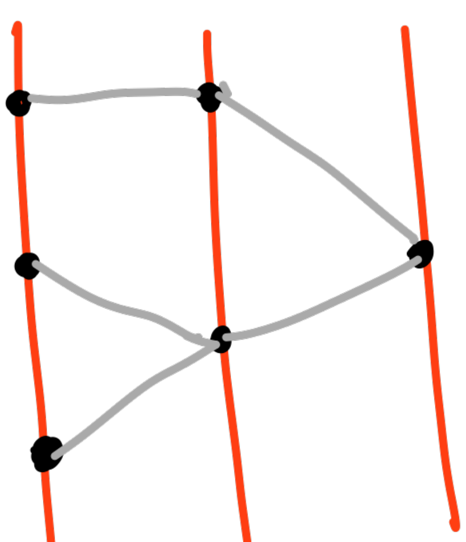
$\circ$	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$	σ
$(0,0)$	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$	σ
$(1,0)$	$(1,0)$	$(2,0)$	$(0,0)$	$(1,1)$	$(2,1)$	$(0,1)$	$(1,2)$	$(2,2)$	$(0,2)$	σ
$(2,0)$	$(2,0)$	$(0,0)$	$(1,0)$	$(2,1)$	$(0,1)$	$(1,1)$	$(2,2)$	$(0,2)$	$(1,2)$	σ
$(0,1)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$	$(0,0)$	$(1,0)$	$(2,0)$	σ
$(1,1)$	$(1,1)$	$(2,1)$	$(0,1)$	$(1,2)$	$(2,2)$	$(0,2)$	$(1,0)$	$(2,0)$	$(0,0)$	σ
$(2,1)$	$(2,1)$	$(0,1)$	$(1,1)$	$(2,2)$	$(0,2)$	$(1,2)$	$(2,0)$	$(0,0)$	$(1,0)$	σ
$(0,2)$	$(0,2)$	$(1,2)$	$(2,2)$	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	σ
$(1,2)$	$(1,2)$	$(2,2)$	$(0,2)$	$(1,0)$	$(2,0)$	$(0,0)$	$(1,1)$	$(2,1)$	$(0,1)$	σ
$(2,2)$	$(2,2)$	$(0,2)$	$(1,2)$	$(2,0)$	$(0,0)$	$(1,0)$	$(2,1)$	$(0,1)$	$(1,1)$	σ
σ	σ	σ	σ	σ	σ	σ	σ	σ	σ	$\sum g$

$\Downarrow$

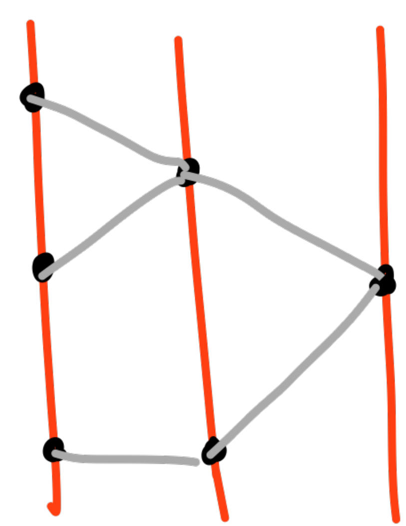
$\mathrm{Vec}(S_3 \times \mathbb{Z}/3)$

$\cong$ Morita

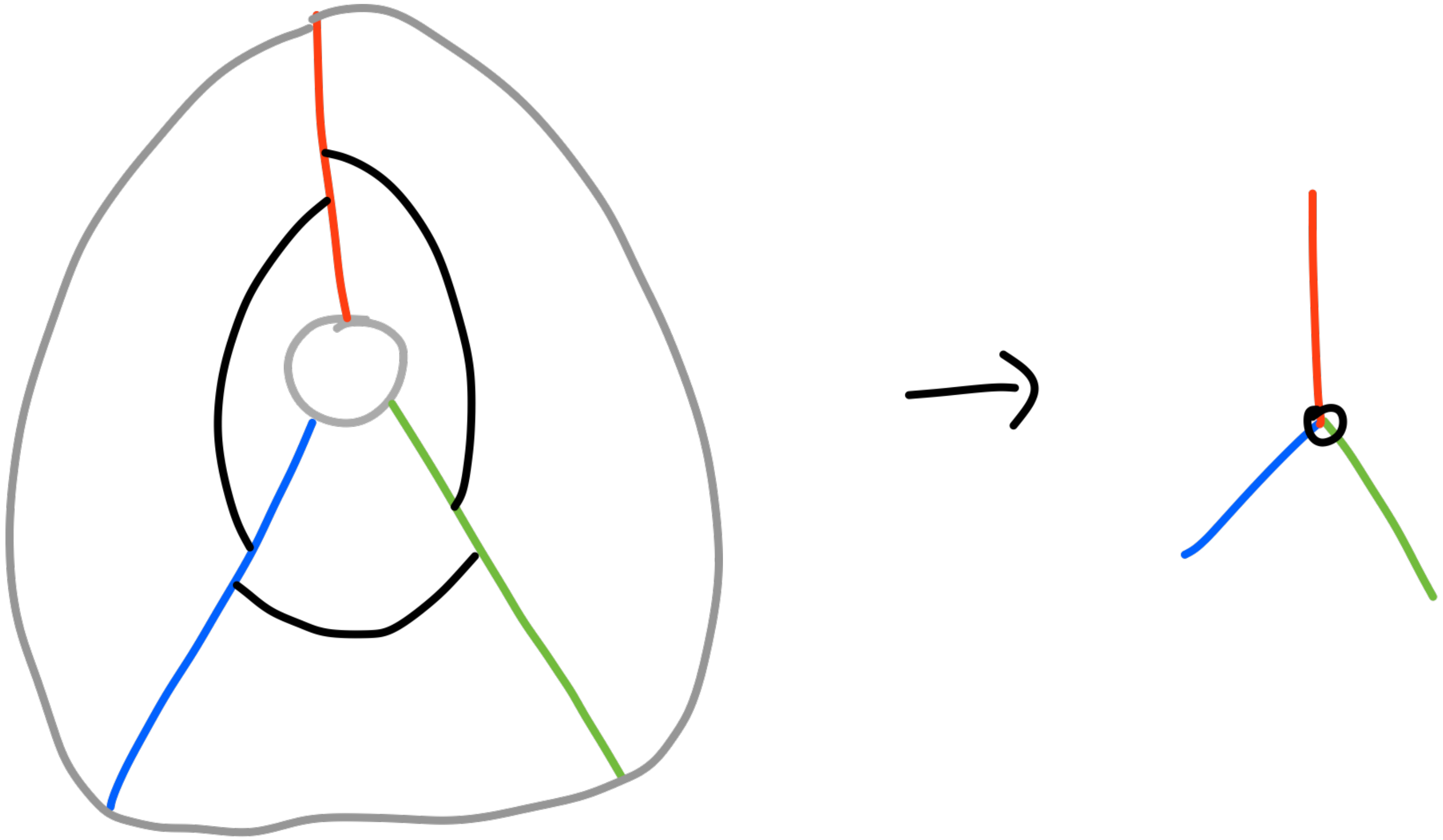
$\mathrm{TY}(\mathbb{Z}/3 \times \mathbb{Z}/3, \chi, 1)$

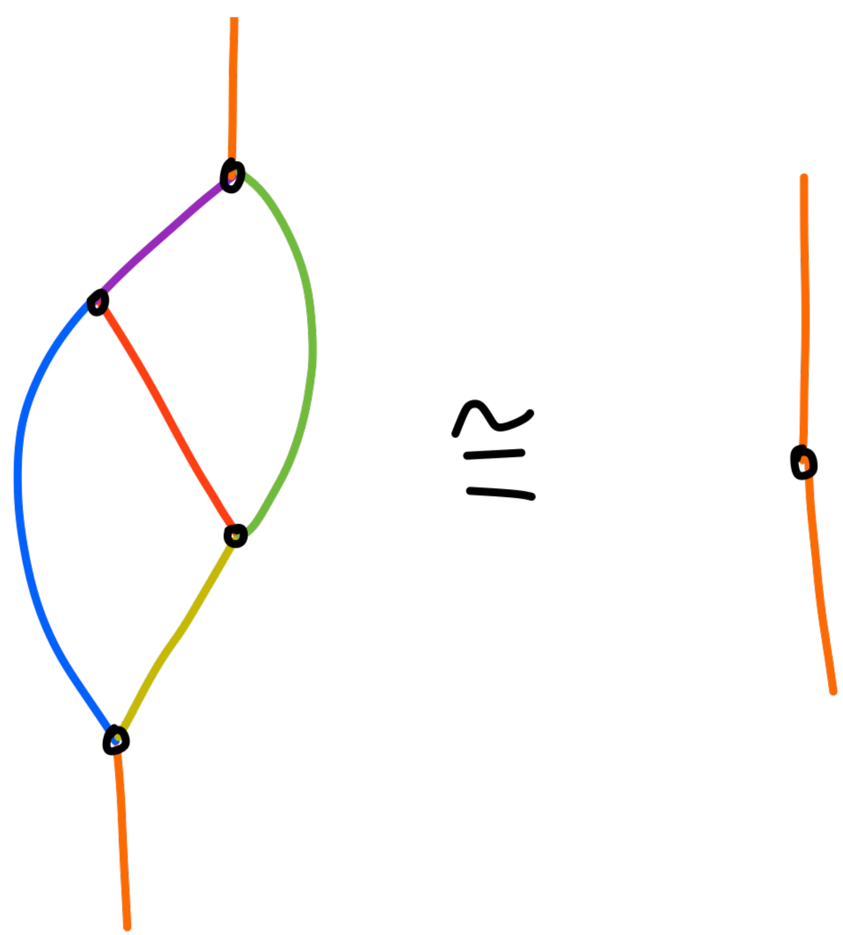


$\xrightarrow{\sim}$



$\Rightarrow$





First ENO obstruction

Vanishes for $\mathbb{Z}/p$

Vanishes for S_3

· Generalised tube algebras let you compute many things by identifying matrix algebras

- Domain wall fusion
- Vertical & horizontal defect fusion
- Defect & domain wall associators
- obstruction to extension

⋮

arXiv : 1806.01279 ($\mathbb{Z}/p$ domain walls with D. Barter, C. Jones)
1810.09469 ($\mathbb{Z}/p$ defects with D. Barter, C. Jones)
1901.08069 ($\mathbb{Z}/p$ obstructions with D. Barter)
1907.XXXXX (S_3 with D. Barter)