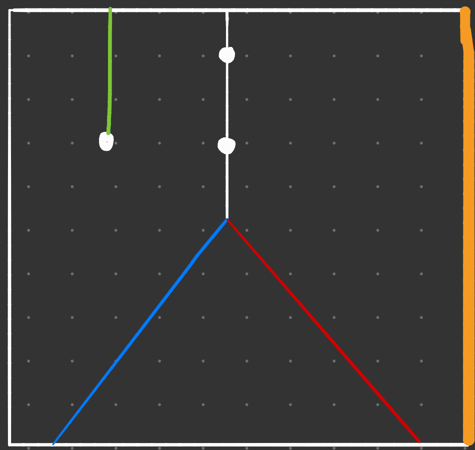
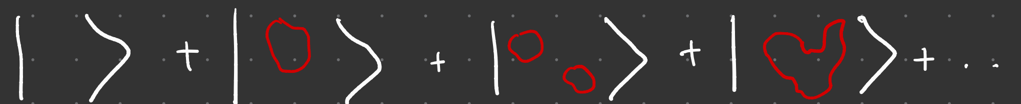
Defects in topological phases.

- 1907.06692 w/ D. Barter
- 1901.08069 "
- 1810.09469 w/ D. Barter & C. Jones.
- 1806.01279 " "

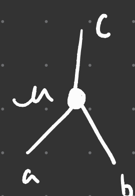


String net models

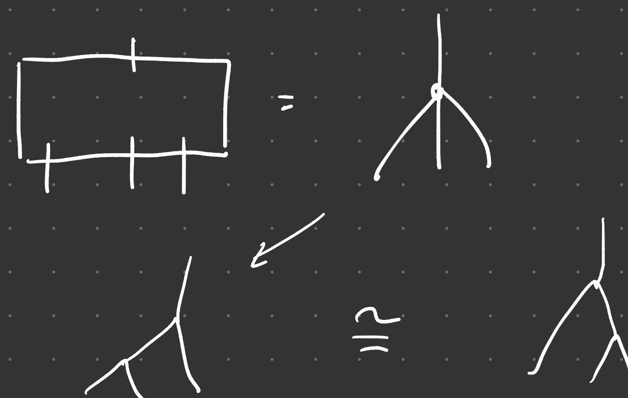
- Playing with pictures.

$|\text{ground state}\rangle \sim$ 

1 :



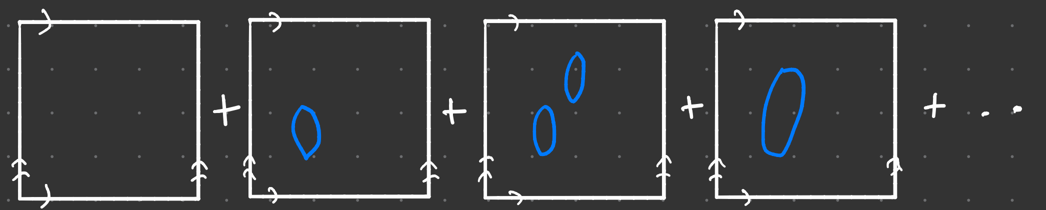
: V_{ab}^{c}

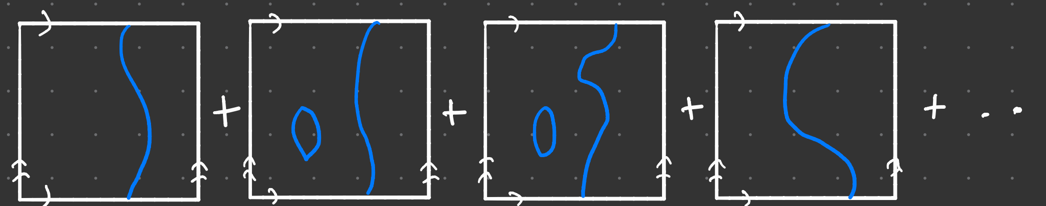


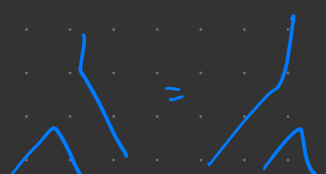
Example : Vec($\mathbb{Z}_2$) (Toric code)

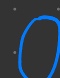
String types : 

fusion rules 

$|\psi_1\rangle =$ 

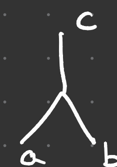
$|\psi_2\rangle =$ 





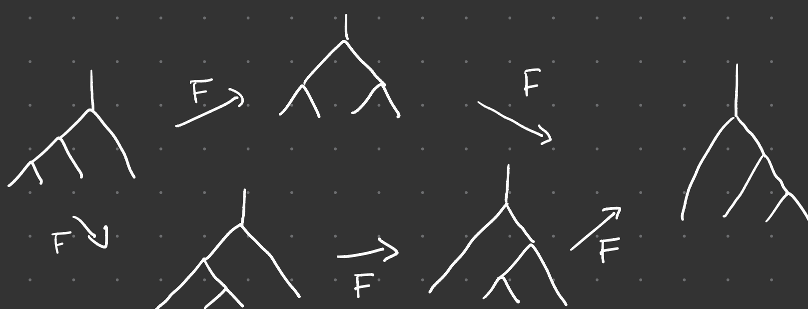
$\vdots$

(Skeletal) Fusion category.

- Finite set of simple objects : $\mathrm{ob}\,\mathcal{C} = \{1, a, b, c, \dots\}$
- Fusion rules : $a \otimes b = \sum_c N_{ab}^c \, c \equiv$ 

$\leftarrow V_{ab}^c$
- Local relations :  $= F$ 

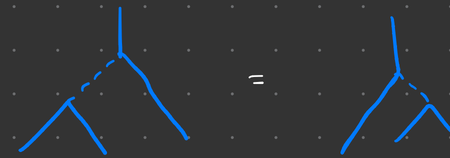
$a \otimes b \cong b \otimes a$.

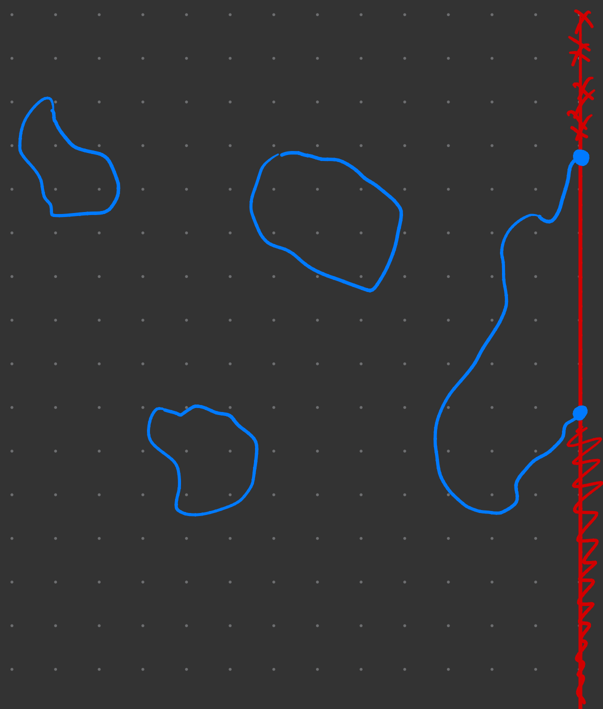


$$\boxed{F^2 = F^3}$$

Boundaries

Toric code example $(\mathrm{Vec}(\mathbb{Z}_2))$





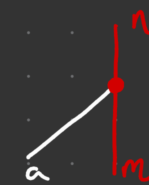
Vacuum

$$(\mathcal{M}\cdot\mathcal{N})\cdot\mathcal{P} \cong \mathcal{M}(\mathcal{N}\cdot\mathcal{P})$$

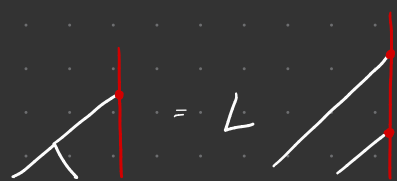
Left $\mathcal{C}$-module category : $\mathcal{C} \curvearrowright \mathcal{M}$

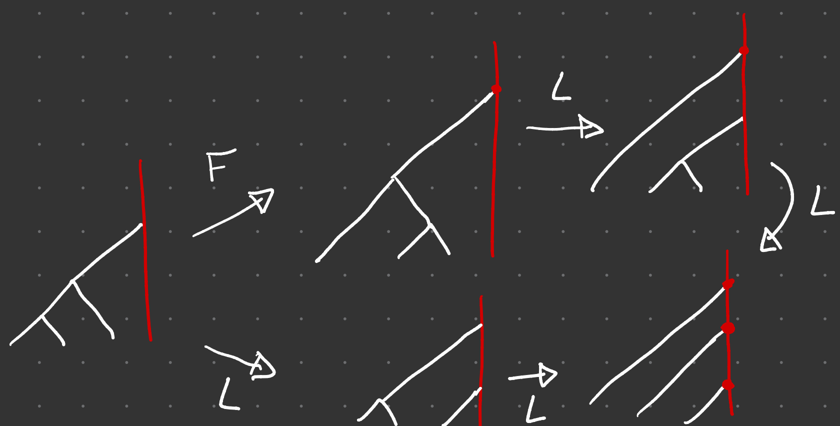
FC bulk.

- Set of objects $\{m, n, \ldots\}$
- Left $\mathcal{C}$ action $a \triangleright m = \sum \tilde{N}_{am}^{n}\, n$



$: V_{am}^{n}$

- 



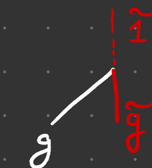
$L^2 = FLF$

Solutions

• $\underline{\mathcal{M}} = \underline{\mathcal{C}}$ forgetting the fusion structure.

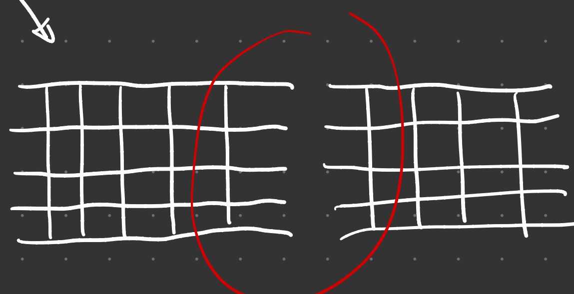
Toric code (Vec $\mathbb{Z}_2$) $= \{ 1, g \mid g \otimes g = 1 \}$.

$$\mathcal{M}_{\text{SMOOTH}} = \{ \tilde{1}, \tilde{g} \} \qquad g \triangleright \tilde{g} = \tilde{1}$$

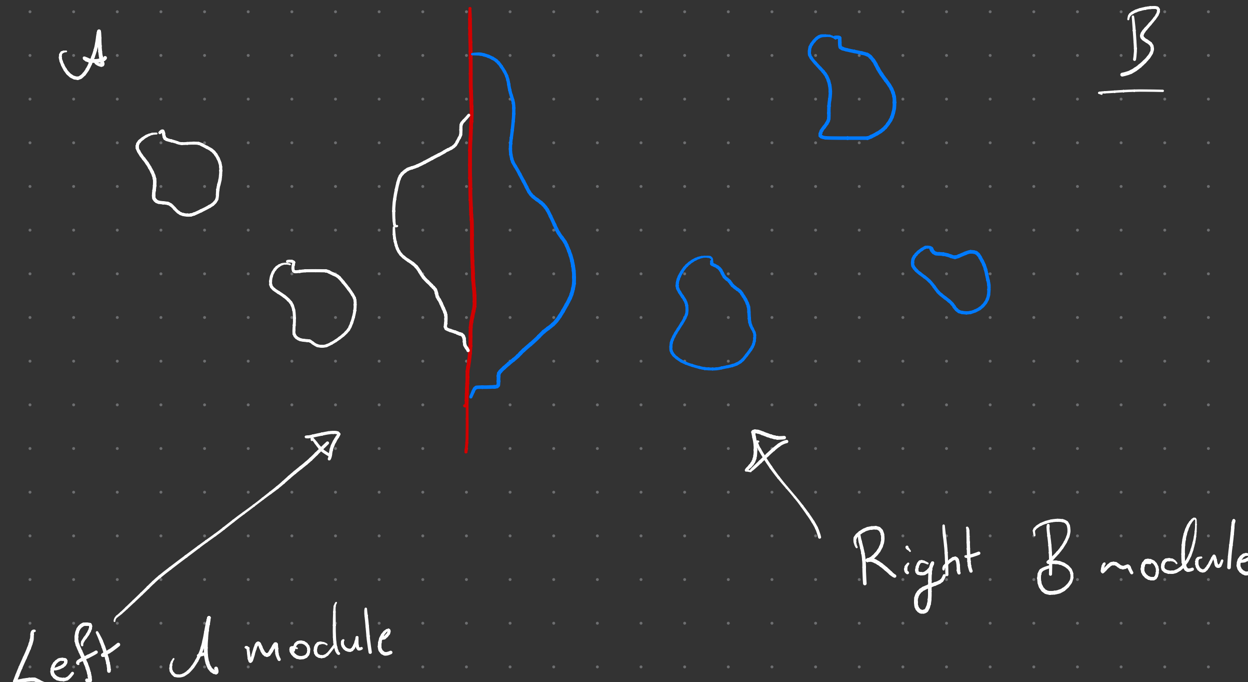


$$\mathcal{M}_{\text{ROUGH}} = \{ * \} \qquad g \triangleright * = *$$

Lattice model:



Interfaces



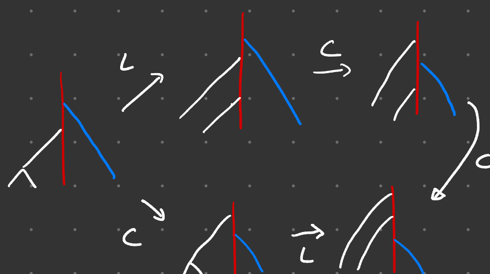
$$\underline{\mathcal{A}} \curvearrowright M \curvearrowleft \underline{\mathcal{B}}$$

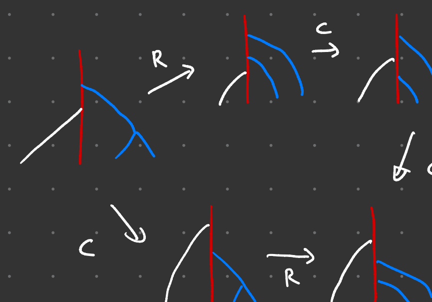
Bimodule category

$$\mathcal{A} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{B}$$

$$\boxed{c^2 L = L c}$$

$$\boxed{c^2 R = R c}$$

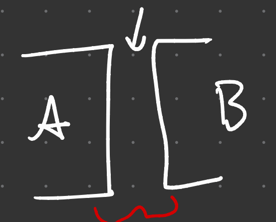




Solutions

$A \times B^{op}$

Always exist : T : $\{(a, b),\ a \in A,\ b \in B\}$.

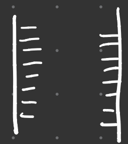
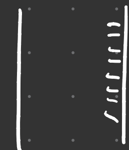
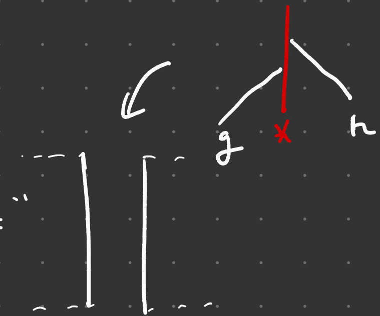


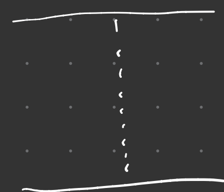
if $A = B$ $\quad X_1$: $\{a \mid a \in A\}$.

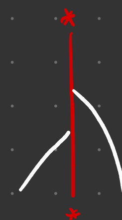
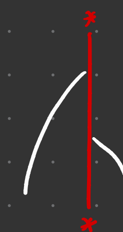
Identity domain wall

$M = A$

Example : Vec $\mathbb{Z}_2$

T "="  , L "="  , R "="  , F_0 "=" 

X_1 ; 

F_1 :  $= (-1)$ 

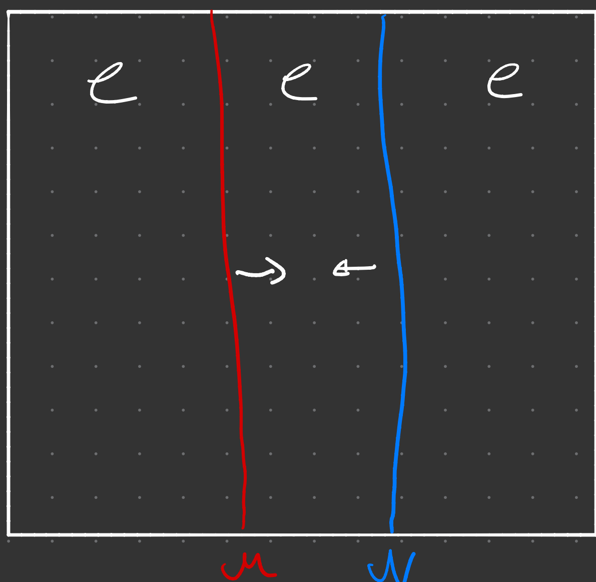
"Fusing" Domain walls

$\downarrow$

Brauer Picard Ring

$$\mathcal{A} \curvearrowright \mathcal{M} \curvearrowleft \mathcal{B} \curvearrowright \mathcal{N} \curvearrowleft \mathcal{E}$$

$$\underbrace{\mathcal{A} \curvearrowright \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \curvearrowleft \mathcal{E}}$$



$$\mathcal{E} \curvearrowright \underbrace{\mathcal{M} \otimes_{\mathcal{E}} \mathcal{N}}_{\text{Composite domain wall}} \curvearrowleft \mathcal{E}$$

$$\boxed{\mathcal{M} \otimes_{\mathcal{E}} \mathcal{N} = \bigoplus_{P} N_{\mathcal{M}\mathcal{N}}^{P} \cdot P}$$

Aside: Representations of a category.

$$F : \mathcal{C} \longrightarrow \mathrm{Vec}$$

ob : $\mathbb{C}^d$

Hom (V, W)

$L : V \to W$.

$F_a : V_a$

$F(f : a \to b) : \quad L_f : V_a \longrightarrow V_b$

$$F(f) \circ F(g) = F(f \circ g)$$

$\underline{G}$: $\mathrm{ob}\,\underline{G} = \{ * \}$

$$\mathrm{Hom}(*,*) = \mathbb{C}G$$

$$\begin{tikzcd} * \arrow[r, "g"] \arrow[dr, "h \circ g = hg"'] & * \arrow[d, "h"] \\ & * \end{tikzcd}$$

$$F : \underline{G} \longrightarrow \mathrm{Vec}$$

$$F_* = \mathbb{C}^d$$

$$\begin{array}{ccccc} F(g) \circ F(h) & = & F(g \circ h) & = & F(gh) \\ \| & & & & \| \\ M_g M_1 & & & & M_{gh} \end{array}$$

Category of $\text{rep}^{\underline{n}}$s : $[\mathscr{C}, \mathrm{Vec}]$

obj $\underline{\underline{\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}}}$: $\{ (m, n, \eta : (\underline{m \triangleleft b}, \underline{n}) \cong (m, \underline{b \triangleright n}) \}$.

$\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \simeq \underline{\mathrm{Fun}_{\mathcal{B}}(\mathcal{M}^{op}, \mathcal{N})}$

Ladder category : objects $\{(m,n) \mid m \in \mathcal{M}, n \in \mathcal{N}\}$.

morphisms : Ladder diagrams

$$\boxed{\text{Thm}: \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \simeq [\mathcal{L}ad(\mathcal{M},\mathcal{N})^{op}, \mathrm{Vec}] \simeq \mathrm{Kar}(\mathcal{L}ad(\mathcal{M},\mathcal{N}))}$$

Proof : See 1806.01279

Computing representations: Karoubi envelope.

$\underline{\mathrm{Kar}}(\mathcal{C})$:

$\mathrm{S}||$

$\underbrace{[\mathcal{C}^{op}, \mathrm{Vec}]}$

objects: $(A \in \mathcal{C},\ i : A \to A)$.

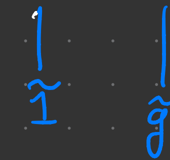
$i \circ i = i$

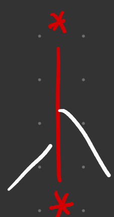
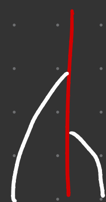
$f \in \mathcal{C}$ $\quad i \circ f = f = f \circ i'$

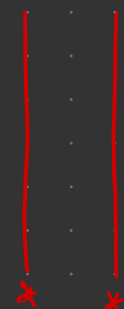
$f : (A, i) \longrightarrow (B, i')$

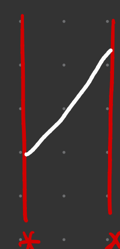
In summary, we can compute the "fusion" of domain walls by finding idempotent ladders.

Example: $\mathrm{Vec}\,\mathbb{Z}_2$. $F_1 \underset{e}{\otimes} F_1 \cong X_1$



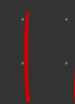
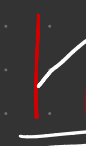
 $= (-1)$ 





M_1

$M_g \; : \; M_g^2 = M_1$

$I_\pm = \frac{1}{2}\Big($  $\pm$  $\Big).$

 $: I_- \to I_+$

$g \triangleright I_- = I_+$

	T	L	R	F_0	X_1	F_1
T	$2T$	T	$2R$	R	T	R
L	$2L$	L	$2F_0$	F_0	L	F_0
R	T	$2T$	R	$2R$	R	T
F_0	L	$2L$	F_0	$2F_0$	F_0	L
X_1	T	L	R	F_0	X_1	F_1
F_1	L	T	F_0	R	F_1	X_1

Brauer Picard Group

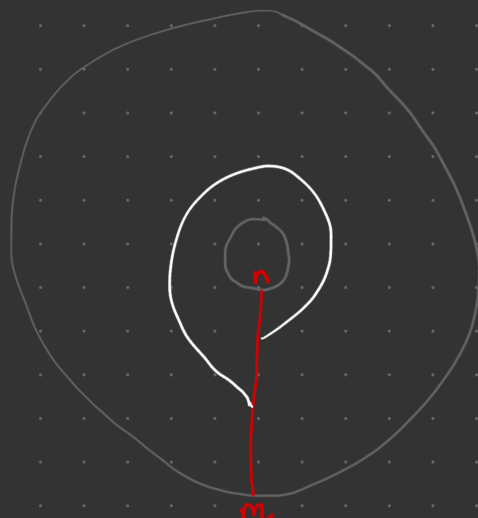
Point Defects

So far, we modified the rules at a line. What about at a point?

→ Anyons
→ "twists"
→ ⋮

Tube category

Hom(m, n) :

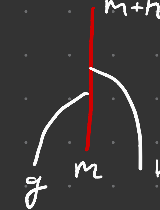


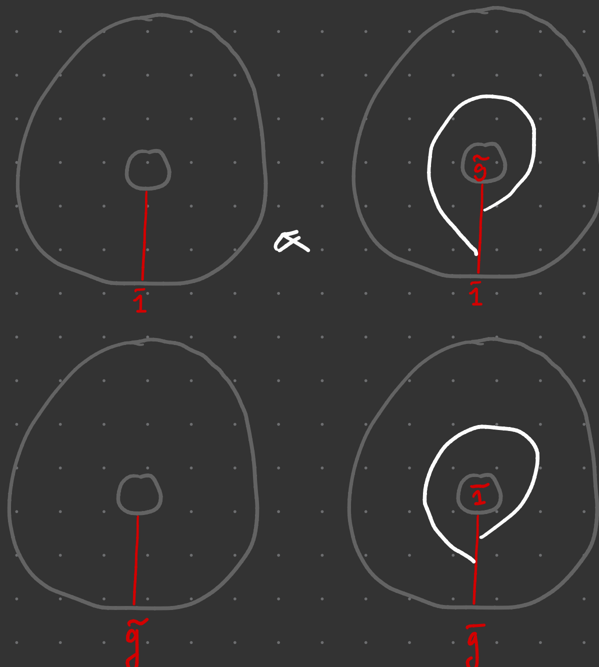
composition

L "=" 

Example. $\mathrm{Vec}(\mathbb{Z}_2)$

L: $\{\tilde{1}, \tilde{g}\}$



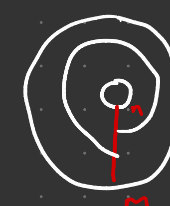


$\Rightarrow$ 1 type of point excitation

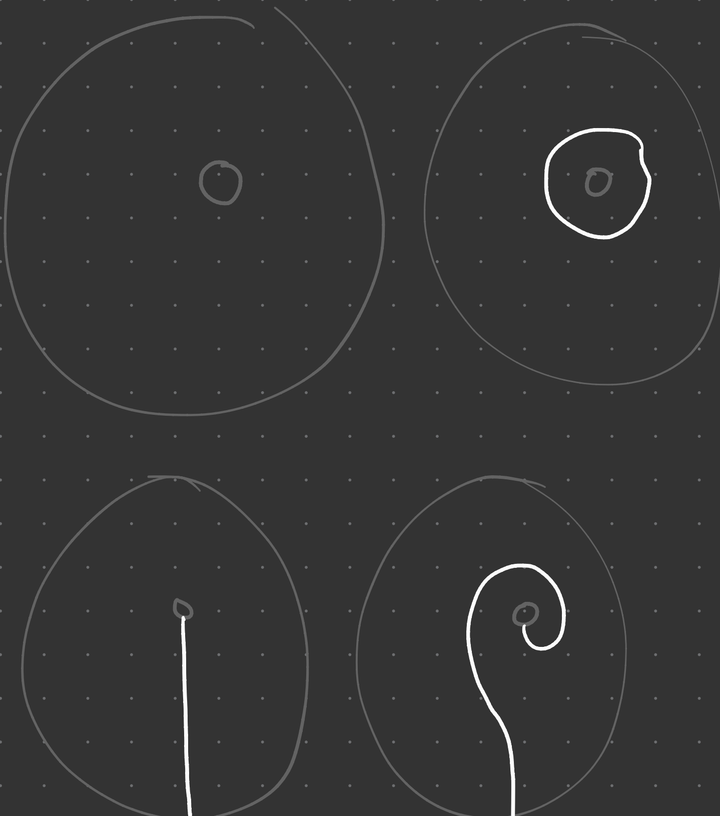
Toric code

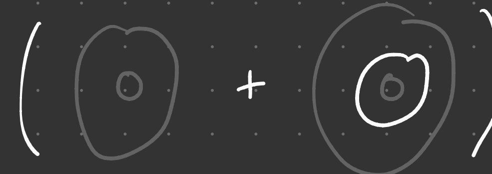
Physically: 

$\mathrm{Kar}(\mathrm{Tub})$: ob (m, n)

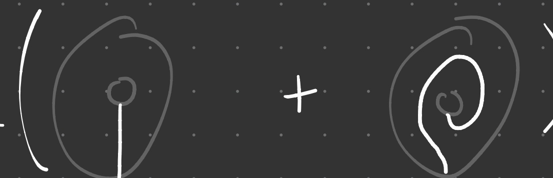
Mor 

Vec($\mathbb{Z}_2$)



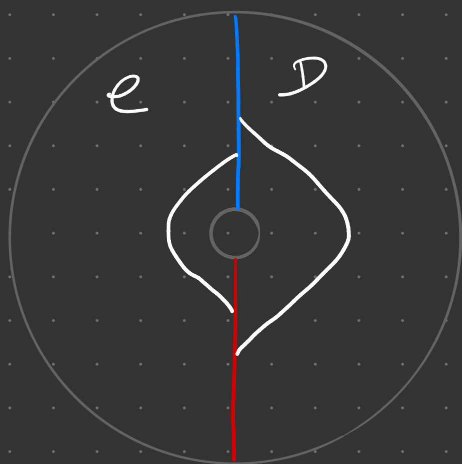
$1 = \frac{1}{2}$ 

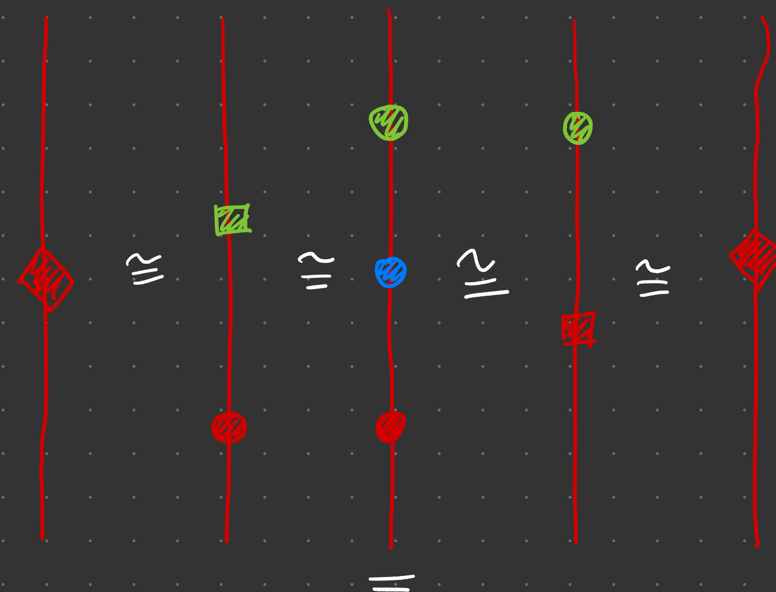
$e = \frac{1}{2}$ 

$m = \frac{1}{2}$ 

$em = \frac{1}{2}$ 

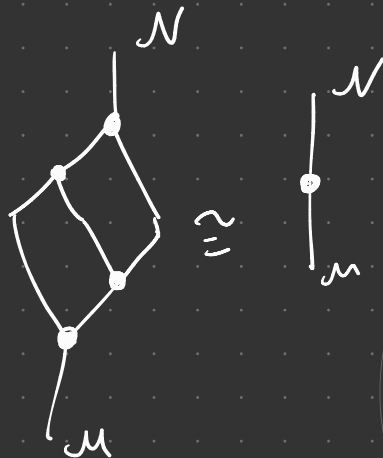
More tube categories.

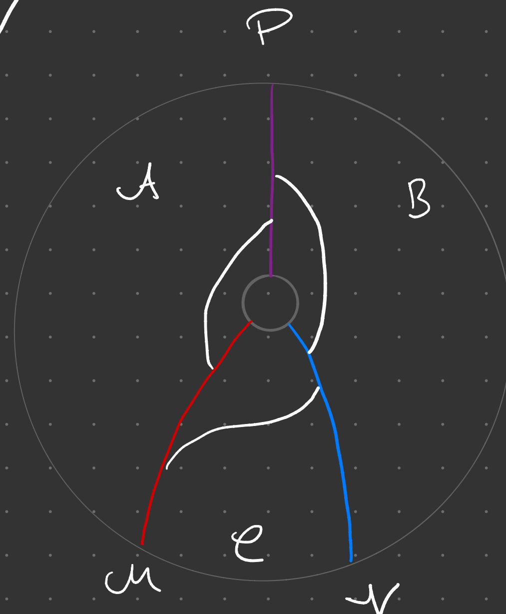


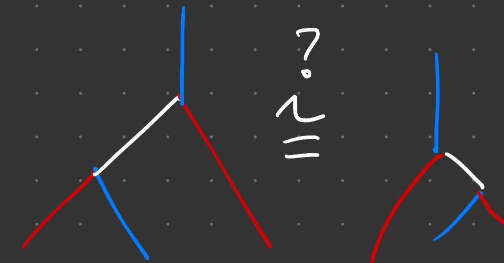


$\underline{\mathrm{End}(\mathcal{M})}$

More tube categories.







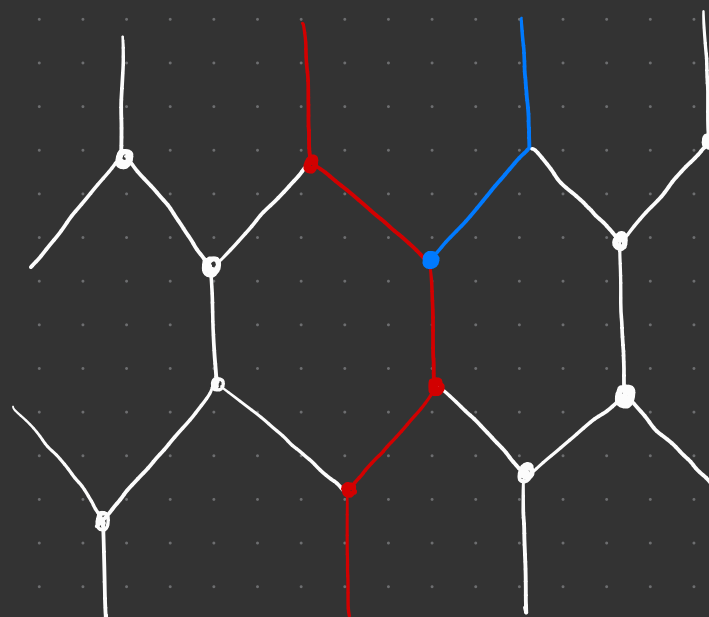
$\rightsquigarrow \; F : \mathcal{M} \otimes_{\mathcal{C}} \mathcal{N} \to \mathcal{P}$

$\rightsquigarrow$

Gauging domain walls

String net models.

Draw network of bimodules



Hamiltonian designed to project to a given point defect

Summary

→ Generalised tube algebras $\rightsquigarrow$ Point defects.

→ Representations of categories, via Karoubi gives way to compute many propeties.

1907.06692

1901.08069

1810.09469

1806.01279

Thank you!